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Discrete Symmetries as Automorphisms of the Proper Poincaré Group

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We present a consistent approach to finding discrete transformations in representation spaces of the proper Poincaré group. To this end we establish a correspondence between involutory automorphisms of the group and the discrete transformations. Such a correspondence allows us to describe the action of discrete transformations on arbitrary spin-tensor fields without any use of relativistic wave equations. Extending the proper Poincaré group by the discrete transformations, we construct explicitly fields carrying corresponding irreps.

KEY WORDS: Poincaré group; discrete symmetries; relativistic wave equations.

1. INTRODUCTION

It is well known that Lorentz transformations in Minkowski space are divided into continuous and discrete ones. Transformations that can be obtained continuously from the identity form the proper Poincaré group. A classification of irreducible representations (irreps) of the Poincaré group was given by Wigner (1939) (see also Barut and Raczka, 1977; Kim and Noz, 1986; Mackey, 1968; Ohnuki, 1988; Tung, 1985). In fact, the representation theory of the proper Poincaré group provides us only by continuous transformations in representation spaces. At the same time, a regular way to describe discrete transformations in such spaces on the ground of purely group-theoretical considerations does not exist. Moreover, it turns out that there is no one-to-one correspondence between the set (P, T) of discrete transformations in Minkowski space and a set of discrete transformations

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in representation spaces. The latter set is wider than the former one (it includes P, T, C, T_w).[5]

As a rule, finding discrete transformations in the representation spaces demands an analysis of corresponding wave equations, and has, in a sense, heuristic character. Besides, the possibility to have different wave equations (and with different symmetries) for particles with the same spin results in a certain "fuzziness" of the definition of discrete transformations in the representation spaces (see, e.g., Benn and Tucker, 1981; Lee and Wick, 1966). All that stresses the lack of a regular approach to the definition of such discrete transformations and, therefore, creates an uncertainty for using discrete transformations as symmetries. More detailed consideration have led Lee and Wick (1966) to the conclusion that "the situation is clearly an unsatisfactory one from a fundamental point of view."

Attempts to define discrete transformations in representation spaces without appealing to any relativistic wave equations or model assumptions have a long history. In particular, some features of discrete transformations were studied on the base of their commutation relations with generators of the Poincaré group (Lee and Wick, 1966; Shirokov, 1958, 1960; Wigner, 1964). One ought to mention also the identification of discrete transformations with (anti)automorphisms of the algebra of observables (Bogolyubov *et al.*, 1990) and the consideration of an action of discrete transformations in terms of the operators of second quantization (see, e.g., Peskin and Schroeder, 1995; Weinberg, 1995). These approaches allow one to avoid straightforward definition of discrete transformations as symmetry ones of relativistic wave equations, but in any case one uses properties of the Dirac equation solutions to cancel the residuary ambiguity. Thus, the problem of an explicit construction of discrete transformations in representation spaces remains still open.

In the present work, we offer the consistent approach to constructing discrete transformations. This approach is completely based on the representation theory of the proper Poincaré group. Our consideration contains two key points.

First, we introduce a scalar field on the proper Poincaré group. This field carries representations with all possible spins and depends on coordinates x on Minkowski space and coordinates z on the Lorentz group. The latter coordinates describe spinning degrees of freedom. Some of discrete transformations affect only space-time coordinates x and some of them affect only spin coordinates z. Using the scalar field we get a possibility to describe "nongeometrical" transformations (ones that leave space-time coordinates x unchanged, in particular, the charge conjugation) on an equal footing with reflections in Minkowski space.

[5] There are three different transformations related to the change of the sign of time: time reflection T considered in detail by Gel'fand *et al.* (1963), Wigner time reversal T_w (Wigner, 1932), and Schwinger time reversal T_{sch} (Schwinger, 1951; Umezava *et al.*, 1954).

Expanding the scalar field in powers of z, we obtain conventional spin-tensor fields as corresponding coefficient functions.

Second, we identify discrete transformations with involutory automorphisms of the proper Poincaré group.

It is known that there are two types of the automorphisms. An inner automorphism of a group G can be presented in the form $g \to g_0 g g_0^{-1}$, where $g_0 \in G$. All other automorphisms are called outer ones. The outer automorphisms of the proper Poincaré group can't be reduced to continuous transformations of the group, they correspond to reflections of coordinate axes or to dilatations. A connection between some discrete transformations and outer automorphisms was mentioned by Gel'fand *et al.* (1963), Gitman and Shelepin (2001), Kuo (1971), Michel (1964), and Silagadze (1992). In particular, study by Kuo (1971) contains an idea that outer automorphisms of internal-symmetry groups may correspond to discrete (possibly broken) symmetries. In this context, one ought to point out the work of Gel'fand *et al.* (1963), where an outer automorphism of the Lorentz group was considered as a starting point to define space reflection.

Studying involutory (both outer and inner) automorphisms of the proper Poincaré group, we describe all discrete transformations and present their action on arbitrary spin-tensor fields without appealing to any relativistic wave equations.

One has to mention a discussion in the literature about the sign of the mass term in relativistic wave equations for half-integer spins (see, e.g., Ahluwalia, 1996; Barut and Ziino, 1993; Brana and Ljolje, 1980; Dvoeglazov, 1996; Markov, 1964). We apply our approach to present a solution for such a problem.

The paper is organized as follows.

In Section 2 we show that outer involutory automorphisms of the Poincaré group are generated by reflections in Minkowski space. Thus, we establish one-to-one correspondence between such automorphisms and reflections.

In Section 3 we introduce the scalar field on the proper Poincaré group and derive transformations of the field under outer and inner automorphisms. As a consequence we find the action of all the discrete transformations (including space and time reflections, charge conjugation, and time reversal) on such a field.

In Section 4, decomposing the scalar field in powers of z, we obtain the action of discrete transformations on conventional spin-tensor fields.

In Section 5 we derive transformations for generators of the Poincaré group and for some other operators under the automorphisms. That allows us to study in detail all the discrete transformations. In particular, we discuss a relation between Wigner and Schwinger time reversals.

In Section 6 we extend the Poincaré group by the discrete transformations and describe characteristics of irreps of the extended group.

In Section 7–9 we construct explicitly massive and massless fields with different characteristics corresponding to the discrete transformations. We establish a relation between our approach and conventional theory of relativistic wave equations.

Finally we classify solutions of relativistic wave equations for arbitrary spins with respect to representations of the extended Poincaré group.

2. REFLECTIONS IN MINKOWSKI SPACE AND OUTER AUTOMORPHISMS OF THE PROPER POINCARÉ GROUP

Here we are going to demonstrate how discrete transformations in Minkowski space can generate some outer involutory automorphisms of the proper Poincaré group. Recall that Poincaré group transformations

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}, \tag{2.1}$$

in Minkowski space ($\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, $x = (x^{\mu}, \mu = 0, 1, 2, 3)$) are defined by pairs (a, Λ), where $a = (a^{\mu})$ is an arbitrary vector and the matrix $\Lambda \in O(3, 1)$. They obey the composition law

$$(a_2, \Lambda_2)(a_1, \Lambda_1) = (a_2 + \Lambda_2 a_1, \Lambda_2 \Lambda_1). \tag{2.2}$$

Any matrix Λ can be presented in one of the four forms: Λ_0, $\Lambda_s\Lambda_0$, $\Lambda_t\Lambda_0$, $\Lambda_s\Lambda_t\Lambda_0$. Here $\Lambda_0 \in SO_0(3, 1)$, where $SO_0(3, 1)$ is a connected component of $SO(3, 1)$, and matrices $\Lambda_s = \mathrm{diag}(1, -1, -1, -1)$, $\Lambda_t = \mathrm{diag}(-1, 1, 1, 1)$ correspond to space reflection P and time reflection T. Then $PT = I_x = \Lambda_s\Lambda_t$. Pairs (a, Λ_0) with the composition law (2.2) form the group $M_0(3, 1)$, which is a semidirect product of the translation group $T(4)$ and the group $SO_0(3, 1)$.

Acting by Λ_s on the equality $x' = \Lambda_0 x + a$, we obtain

$$\Lambda_s x' = \Lambda_s \Lambda_0 \Lambda_s^{-1} \Lambda_s x + \Lambda_s a, \quad \text{or} \quad \overline{x}' = \overline{\Lambda}_0 \overline{x} + \overline{a},$$

where

$$\overline{x} = \Lambda_s x = (x^0, -x^k), \quad \overline{a} = \Lambda_s a = (a^0, -a^k), \quad \overline{\Lambda}_0 = \Lambda_s \Lambda_0 \Lambda_s^{-1} = \left(\Lambda_0^T\right)^{-1}.$$

In a similar manner, using the operations T and I_x, we obtain finally that P, T, I_x generate three outer involutory automorphisms of $M_0(3, 1)$,

$$\begin{aligned} P &: (a, \Lambda_0) \to \left(\overline{a}, \left(\Lambda_0^T\right)^{-1}\right); \\ T &: (a, \Lambda_0) \to \left(-\overline{a}, \left(\Lambda_0^T\right)^{-1}\right); \\ I_x &: (a, \Lambda_0) \to (-a, \Lambda_0). \end{aligned} \tag{2.3}$$

Notice that P and T generate the same automorphism of the group $SO_0(3, 1)$.

Consider now a group $M(3, 1)$, which is an universal covering group for $M_0(3, 1)$. $M(3, 1)$ is the semidirect product of $T(4)$ and $SL(2, C)$ and will be called further the proper Poincaré group. (The extension of the proper Poincaré group by the space reflection will be called below the improper Poincaré group.) It is known that there is one-to-one correspondence between any vectors v from

Minkowski space and 2×2 Hermitian matrices[6] V (see, e.g., Barut and Raczka, 1977; Buchbinder and Kuzenko, 1995; Streater and Wightman, 1964):

$$v^\mu \leftrightarrow V = v^\mu \sigma_\mu, \quad v^\mu = \frac{1}{2}\mathrm{Tr}(V\bar{\sigma}^\mu). \tag{2.5}$$

Proper Poincaré transformations $x' = \Lambda_0 x + a$ can be rewritten in new terms as

$$X' = UXU^\dagger + A, \tag{2.6}$$

where $X = x^\mu \sigma_\mu$, $A = a^\mu \sigma_\mu$, and $U \in SL(2, C)$ (two different matrices $\pm U$ correspond to one matrix Λ_0). Elements of $M(3, 1)$ are given by pairs (A, U) with the composition law

$$(A_2, U_2)(A_1, U_1) = (U_2 A_1 U_2^\dagger + A_2, U_2 U_1). \tag{2.7}$$

Space reflection takes $x = (x^0, x^k)$ into $\bar{x} = (x^0, -x^k)$, or in terms of $X = x^\mu \sigma_\mu$,

$$P : X \to \bar{X} = \bar{x}^\mu \sigma_\mu = x^\mu \bar{\sigma}_\mu.$$

Using the relation $\bar{X} = \sigma_2 X^T \sigma_2$ and the identity $\sigma_2 U \sigma_2 = (U^T)^{-1}$, we obtain

$$\bar{X}' = (U^\dagger)^{-1} \bar{X} U^{-1} + \bar{A} \tag{2.8}$$

as a consequence of (2.6). Thus, $\bar{X}$ is transformed by means of the element $(\bar{A}, (U^\dagger)^{-1})$ of $M(3, 1)$. The relation

$$P : (A, U) \to (\bar{A}, (U^\dagger)^{-1}) \tag{2.9}$$

defines an outer involutory automorphism of $M_0(3, 1)$. In a similar manner, we obtain automorphisms of $M(3, 1)$ that are generated by T, I_x,

$$T : (A, U) \to (-\bar{A}, (U^\dagger)^{-1}); \tag{2.10}$$

$$I_x : (A, U) \to (-A, U). \tag{2.11}$$

The automorphisms that correspond to P and T exhaust all the outer involutory automorphisms of the proper Poincaré group in the following sense. Any outer involutory automorphism can be presented as a composition of these two automorphisms and an inner automorphism of the group.[7] In particular, the automorphism

[6] We use two sets of 2×2 matrices $\sigma_\mu = (\sigma_0, \sigma_k)$ and $\bar{\sigma}_\mu = (\sigma_0, -\sigma_k)$,

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.4}$$

[7] The Poincaré group $M(3, 1)$ is a semidirect product of the Lorentz group $SL(2, C)$ and the group of four-dimensional translations $T(4)$. Any outer automorphism of $SL(2, C)$ is a product of involutory automorphism $U \to (U^\dagger)^{-1}$ and of an inner automorphism (Gel'fand *et al.*, 1963). Outer automorphisms of the translation group are generated by the dilatations $x^\mu \to cx^\mu$, $c \neq 0, 1$, and are involutory only at $c = -1$. Outer automorphisms of $SL(2, C)$ and $T(4)$ generate the following outer automorphisms of the Poincaré group: $(A, U) \to (\bar{A}, (U^\dagger)^{-1})$, $(A, U) \to (cA, U)$.

of complex conjugation

$$C : (A, U) \to (\overset{*}{A}, \overset{*}{U}). \tag{2.12}$$

is the product of the outer automorphism (2.9) and the inner automorphism

$$(0, i\sigma_2)(\overline{A}, (U^\dagger)^{-1})(0, -i\sigma_2) = (\overset{*}{A}, \overset{*}{U}). \tag{2.13}$$

One can see from (2.9) and (2.10) that P and T generate the same automorphisms of $SL(2, C)$, namely, $U \to (U^\dagger)^{-1}$, whereas PT generates the identity automorphism of $SL(2, C)$ and an outer automorphism $A \to -A$ of the translation group.

Thus, the discrete transformations in Minkowski space generate outer involutory automorphisms of the proper Poincaré group.[8] In the next section we will see how these automorphisms are related to discrete transformations in representation spaces of the group.

3. AUTOMORPHISMS AND DISCRETE TRANSFORMATIONS IN REPRESENTATION SPACES

The main object of our study here is a scalar field on the proper Poincaré group (Gitman and Shelepin, in press). That field is in fact a generating function for all irreps of the group. First we recall briefly main points of the corresponding technique. It is well known (Barut and Raczka, 1977; Vilenkin, 1968; Zhelobenko and Schtern, 1983) that any irrep of a group G is contained (up to the equivalence) in a decomposition of a generalized regular representation. Consider the left generalized regular representation $T_{\mathrm{L}}(g)$, which is defined in the space of functions $f(h)$, $h \in G$ on the group as

$$T_{\mathrm{L}}(g) f(h) = f'(h) = f(g^{-1}h), \quad g \in G. \tag{3.1}$$

As a consequence of the relation (3.1), we can write

$$f'(h') = f(h), \quad h' = gh. \tag{3.2}$$

Let G be the group $M(3, 1)$ and we use the parametrization by two 2×2 matrices (one Hermitian and another one from $SL(2, C)$), which was described in the previous section. At the same time, using such a parametrization, we choose the following notations:

$$g \leftrightarrow (A, U), \quad h \leftrightarrow (X, Z), \tag{3.3}$$

[8] The reflections P, T can be considered also in terms of fundamental automorphisms of Clifford algebras (Varlamov, Preprint math-ph/0009026).

where A, X are 2×2 Hermitian matrices and $U, Z \in SL(2, C)$. The map $h \leftrightarrow (X, Z)$ creates the correspondence

$$h \leftrightarrow (x, z, \underline{z}), \quad x = (x^\mu), \quad z = (z_\alpha), \quad \underline{z} = (\underline{z}_\alpha),$$
$$\mu = 0, 1, 2, 3, \quad \alpha = 1, 2, \quad z_1\underline{z}_2 - z_2\underline{z}_1 = 1 \tag{3.4}$$

by virtue of the relations

$$X = x^\mu \sigma_\mu, \quad Z = \begin{pmatrix} z_1 & \underline{z}_1 \\ z_2 & \underline{z}_2 \end{pmatrix} \in SL(2, C). \tag{3.5}$$

On the other hand, we have the correspondence $h' \leftrightarrow (x', z', \underline{z}')$,

$$h' = gh \leftrightarrow (X', Z') = (A, U)(X, Z) = (UXU^+ + A, UZ) \leftrightarrow (x', z', \underline{z}'),$$
$$x'^\mu \sigma_\mu = X' = UXU^+ + A \Rightarrow x'^\mu = (\Lambda_0)^\mu{}_\nu x^\nu + a^\mu, \quad \Lambda_0 \leftarrow U \in SL(2, C), \tag{3.6}$$

$$\begin{pmatrix} z'_1 & \underline{z}'_1 \\ z'_2 & \underline{z}'_2 \end{pmatrix} = Z' = UZ \Rightarrow z'_\alpha = U_\alpha{}^\beta z_\beta, \quad \underline{z}'_\alpha = U_\alpha{}^\beta \underline{z}_\beta,$$
$$U = \left(U_\alpha{}^\beta\right), \quad z'_1\underline{z}'_2 - z'_2\underline{z}'_1 = 1. \tag{3.7}$$

Then the relation (3.2) takes the form

$$f'(x', z', \underline{z}') = f(x, z, \underline{z}), \tag{3.8}$$

$$x'^\mu = (\Lambda_0)^\mu{}_\nu x^\nu + a^\mu, \quad \Lambda_0 \leftarrow U \in SL(2, C), \tag{3.9}$$

$$z'_\alpha = U_\alpha{}^\beta z_\beta, \quad \underline{z}'_\alpha = U_\alpha{}^\beta \underline{z}_\beta, \quad z_1\underline{z}_2 - z_2\underline{z}_1 = z'_1\underline{z}'_2 - z'_2\underline{z}'_1 = 1. \tag{3.10}$$

The relations (3.8)–(3.10) admit a remarkable interpretation. We may treat x and x' in these relations as position coordinates in Minkowski space $M(3, 1)/SL(2, C)$ (in different Lorentz reference frames) related by proper Poincaré transformations, and sets $z, \underline{z}$ and $z', \underline{z}'$ may be treated as spin coordinates in these Lorentz frames. They are transformed according to Eq. (3.10). Carrying two-dimensional spinor representation of the Lorentz group, the variables z and $\underline{z}$ are invariant under translations as one can expect for spin degrees of freedom. Thus, we may treat sets $x, z, \underline{z}$ as points in a position-spin space with the transformation law (3.9) and (3.10) under the change from one Lorentz reference frame to another. In this case Eqs. (3.8)–(3.10) present the transformation law for scalar functions on the position-spin space.

On the other hand, as we have seen, the set $(x, z, \underline{z})$ is in one-to-one correspondence with elements of $M(3, 1)$. Thus, the functions $f(x, z, \underline{z})$ are still functions on this group. That is why we often call them scalar functions on the group, remembering that the term "scalar" came from the above interpretation.

We are reminded that different functions of such type correspond to different representations of the group $M(3, 1)$. Thus, the problem of classification of irreps of this group is reduced to the problem of a classification of the scalar functions on position-spin space. It is natural to restrict ourselves by scalar functions that are analytic both in z, $\underline{z}$ and in $\overset{*}{z}$, $\overset{*}{\underline{z}}$ (or, simply speaking, that are differentiable with respect to these arguments). Further, such functions are denoted by $f(x, z, \underline{z}, \overset{*}{z}, \overset{*}{\underline{z}}) = f(x, \mathbf{z})$, $\mathbf{z} = (z, \underline{z}, \overset{*}{z}, \overset{*}{\underline{z}})$. Since matrices U are unimodular, there exist invariant antisymmetric tensors $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$, $\varepsilon^{\dot{\alpha}\dot{\beta}} = -\varepsilon^{\dot{\beta}\dot{\alpha}}$, $\varepsilon^{12} = \varepsilon^{\dot{1}\dot{2}} = 1$, $\varepsilon_{12} = \varepsilon_{\dot{1}\dot{2}} = -1$. Spinor indices are lowered and raised by the help of these tensors,

$$z_\alpha = \varepsilon_{\alpha\beta} z^\beta, \quad z^\alpha = \varepsilon^{\alpha\beta} z_\beta, \quad \overset{*}{z}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}} \overset{*}{z}{}^{\dot{\beta}}, \quad \overset{*}{z}{}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}} \overset{*}{z}_{\dot{\beta}}.$$

The continuous transformations (3.10) do not mix z^α and $\underline{z}^\alpha$ (and their complex conjugate $\overset{*}{z}{}^{\dot{\alpha}}$, $\overset{*}{\underline{z}}{}^{\dot{\alpha}}$). Therefore, scalar functions of the form $f(x, z)$, $f(x, \underline{z})$, $f(x, \overset{*}{z})$, $f(x, \overset{*}{\underline{z}})$ form four invariant (with respect to $M(3, 1)$ transformations) subspaces.

As was demonstrated by Gitman and Shelepin (in press), a standard spin description in terms of multicomponent functions arises under the separation of space and spin variables in the scalar functions. Below we recall how it works.

Since $\mathbf{z}$ is invariant under translations, any function $\phi(\mathbf{z})$ carries a representation of the Lorentz group. Let a function $f(h) = f(x, \mathbf{z})$ can be presented in the form

$$f(x, \mathbf{z}) = \phi^n(\mathbf{z})\psi_n(x), \tag{3.11}$$

where $\phi^n(\mathbf{z})$ form a basis in the representation space of the Lorentz group. Thus, we can decompose the functions $\phi^n(\mathbf{z}')$ of transformed argument $\mathbf{z}' = g\mathbf{z}$ in terms of the functions $\phi^n(\mathbf{z})$,

$$\phi^n(\mathbf{z}') = \phi^l(\mathbf{z}) L_l{}^n(U). \tag{3.12}$$

Then the action of the Poincaré group on a line $\phi(\mathbf{z}) = (\phi^n(\mathbf{z}))$ is reduced to a multiplication by a matrix $L(U)$, where $U \in SL(2, C)$: $\phi(\mathbf{z}') = \phi(\mathbf{z})L(U)$. Comparing decompositions of a function $f'(x', \mathbf{z}') = f(x, \mathbf{z})$ in terms of the transformed basis $\phi(\mathbf{z}')$ and in terms of the initial basis $\phi(\mathbf{z})$,

$$f'(x', \mathbf{z}') = \phi(\mathbf{z}')\psi'(x') = \phi(\mathbf{z})L(U)\psi'(x') = \phi(\mathbf{z})\psi(x),$$

where $\psi(x)$ is a column with components $\psi_n(x)$, we obtain

$$\psi'(x') = L(U^{-1})\psi(x). \tag{3.13}$$

This is the transformation law for tensor fields on Minkowski space. This law can be treated as a representation of the Poincaré group acting in a linear space of tensor fields as follows $T(g)\psi(x) = L(U^{-1})\psi(\Lambda^{-1}(x - a))$. According to (3.12) and (3.13), the functions $\phi(\mathbf{z})$ and $\psi(x)$ are transformed under contragradient representations of the Lorentz group.

Consider now the action of automorphisms in the space of the scalar functions. The automorphisms $g \to IgI^{-1}$ (both inner and outer) generate the following left generalized regular representation transformations of the Poincaré group:

$$T_{\mathrm{L}}(g) \to IT_{\mathrm{L}}(g)I^{-1} \equiv T_{\mathrm{L}}(IgI^{-1}), \tag{3.14}$$

$$f(h) \to If(h) \equiv f(IhI^{-1}). \tag{3.15}$$

Equation (3.15) defines a corresponding mapping of the space of functions $f(h)$ into itself.

Transformations of (A, U) and X under the automorphisms that correspond to space and time reflections are given by Eqs. (2.9)–(2.11). Notice that the composition law of the group is not changed under the automorphisms, therefore (X, Z) is transformed just as (A, U):

$$P : (X, Z) \to (\bar{X}, (Z^{\dagger})^{-1}); \tag{3.16}$$

$$T : (X, Z) \to (-\bar{X}, (Z^{\dagger})^{-1}); \tag{3.17}$$

$$I_x : (X, Z) \to (-X, Z). \tag{3.18}$$

We see that the automorphisms in question correspond to a replacement of arguments of scalar functions $f(h)$ according to Eqs. (3.16)–(3.18).

The replacement

$$Z \xrightarrow{P,T} (Z^{\dagger})^{-1}, \quad \text{or} \quad \begin{pmatrix} z^1 & \underline{z}^1 \\ z^2 & \underline{z}^2 \end{pmatrix} \xrightarrow{P,T} \begin{pmatrix} -\overset{*}{\underline{z}}_{\dot{1}} & \overset{*}{z}_{\dot{1}} \\ -\overset{*}{\underline{z}}_{\dot{2}} & \overset{*}{z}_{\dot{2}} \end{pmatrix} \tag{3.19}$$

corresponds to space and time reflections. The transformation (3.19) maps functions of z^{α} into functions of $\overset{*}{\underline{z}}_{\dot{\alpha}}$. Thus, the space of the scalar functions contains two invariant (with respect to proper Poincaré group transformations and space-time reflections) subspaces of functions of the form $f(x, z, \overset{*}{\underline{z}})$ and $f(x, \underline{z}, \overset{*}{z})$. We denote these two subspaces by V_+ and V_- respectively.

The complex conjugation C affects both the form of scalar functions and maximal set of coordinates on the Lorentz group,

$$C : T(g) \to CT(g)C^{-1} \equiv \overset{*}{T}(g), \quad f(h) \to Cf(h) \equiv \overset{*}{f}(h). \tag{3.20}$$

Therefore, such a transformation takes subspaces V_+ and V_- into one another. The transformation (3.20) of the field $f(h)$ can be identified (Gitman and Shelepin, in press) with the charge conjugation, which interchanges particle and antiparticle fields, see below.

Studying involutory outer automorphisms of the proper Poincaré group and complex conjugation in the space of scalar functions on the group, we have defined three independent discrete transformations (space reflection P, time reflection T, and charge conjugation C) in the representation space of the group. However, there exist two more independent discrete transformations. Indeed, it is easy to see that transformation laws of arguments of functions $f(h)$ under Lorentz rotations and under inner automorphisms are different,

$$(0, U)(X, Z) = (UXU^\dagger, UZ), \tag{3.21}$$

$$(0, U)(X, Z)(0, U^{-1}) = (UXU^\dagger, UZU^{-1}). \tag{3.22}$$

In both cases coordinates x are transformed similarly, thus the action of the inner automorphisms (3.22) in the space of scalar functions $f(x)$ on Minkowski space is reduced to Lorentz rotations. But for functions $f(x, \mathbf{z})$ of general form, the action of inner automorphisms is more complicated.

Suppose an inner automorphism (3.22) corresponds to a discrete transformation, then the conditions $U^2 = e^{i\phi}$ and $\det U = 1$ take place. Diagonal matrices with elements $e^{i\phi/2}$ and matrices of the form

$$U = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad a^2 + bc = -1, \quad U^2 = -1, \tag{3.23}$$

satisfy these conditions. Since the composition of discrete transformations is a discrete transformation as well, the square of product of two different matrices of the form (3.23) must be (up to a phase factor) the identity matrix. The latter requirement reduces (up to a sign) the set of all the matrices (3.23) to the only three matrices

$$i\sigma_1, \ i\sigma_2, \ i\sigma_3.$$

The matrix $U = i\sigma_2$ presents an explicit realization of an inner involutory automorphism

$$(X, Z) \to (\bar{X}^T, (Z^T)^{-1}). \tag{3.24}$$

(Such a realization was already used above, see (2.13).) The automorphism (3.24) change signs of two coordinates x^1, x^3 and does not change x^2 (that correspond to the rotation by the angle π in Minkowski space). It is more convenient to consider a composition of the inner automorphism corresponding to the element $(0, U)$ and the Lorentz rotation corresponding to the element $(0, U^{-1})$, namely

$$(X, Z) \to (X, ZU^{-1}). \tag{3.25}$$

Selecting $U = i\sigma_2$, we obtain the transformation (we denote it by I_z),

$$I_z : (X, Z) \to (X, Z(-i\sigma_2)), \quad \begin{pmatrix} z^1 & \underline{z}^1 \\ z^2 & \underline{z}^2 \end{pmatrix} \to \begin{pmatrix} \underline{z}^1 & -z^1 \\ \underline{z}^2 & -z^2 \end{pmatrix}. \tag{3.26}$$

This transformation maps the spaces of functions $f(x, z, \overset{*}{\underline{z}})$ and $f(x, \underline{z}, \overset{*}{z})$ into one another. In contrast to the charge conjugation (3.20), the transformation I_z replaces arguments only.

Selecting $U = i\sigma_3$ we obtain the transformation

$$I_3 : (X, Z) \to (X, Z(-i\sigma_3)), \quad \begin{pmatrix} z^1 & \underline{z}^1 \\ z^2 & \underline{z}^2 \end{pmatrix} \to \begin{pmatrix} -iz^1 & i\underline{z}^1 \\ -iz^2 & i\underline{z}^2 \end{pmatrix}. \tag{3.27}$$

The transformation associated with $U = i\sigma_1$ is a product of the transformations I_z and I_3.

We see that whereas in Minkowski space there exist only two independent discrete transformations corresponding to outer automorphisms of the Poincaré group, for the scalar field on the group there exist five independent discrete transformations corresponding to both outer and inner automorphisms. Charge conjugation is associated with complex conjugation of the scalar functions, and another four transformations are associated with the following replacements of arguments of the scalar functions:

	x^0	$\mathbf{x}$	z^α	$\overset{*}{z}_{\dot\alpha}$	$\underline{z}^\alpha$	$\underline{\overset{*}{z}}_{\dot\alpha}$
P	x^0	$-\mathbf{x}$	$-\underline{\overset{*}{z}}_{\dot\alpha}$	$\underline{z}^\alpha$	$\overset{*}{z}_{\dot\alpha}$	$-z^\alpha$
I_x	$-x^0$	$-\mathbf{x}$	z^α	$\overset{*}{z}_{\dot\alpha}$	$\underline{z}^\alpha$	$\underline{\overset{*}{z}}_{\dot\alpha}$
I_z	x^0	$\mathbf{x}$	$\underline{z}^\alpha$	$\underline{\overset{*}{z}}_{\dot\alpha}$	$-z^\alpha$	$-\overset{*}{z}_{\dot\alpha}$
I_3	x^0	$\mathbf{x}$	$-iz^\alpha$	$i\overset{*}{z}_{\dot\alpha}$	$i\underline{z}^\alpha$	$-i\underline{\overset{*}{z}}_{\dot\alpha}$

(3.28)

4. DISCRETE TRANSFORMATIONS OF SPIN-TENSOR FIELDS

Decomposing the scalar fields in powers of $\mathbf{z} = (z, \underline{z}, \overset{*}{z}, \underline{\overset{*}{z}})$, we obtain all conventional spin-tensor fields. The latter are coefficient functions in such decompositions and depend on coordinates x on Minkowski space. Thus, we can derive the action of all the discrete transformations on spin-tensor fields.

There is only one type of spinors in nonrelativistic theory (all spinors are subjected to the same transformation law under rotations), and there are two types of spinors (dotted and undotted, which are subjected to different transformation laws under boosts) in relativistic theory. Underlined and nonunderlined z-spinors have different transformation laws under discrete transformations. Thus, taking into account discrete transformations, we should consider four types of spinors: dotted and undotted (or left and right) and underlined and nonunderlined (which allow one to differ particles and antiparticles). In contrast to spin-tensor fields, for the fields on the Poincaré group the use of different types of indices is not necessary, because these indices only duplicate the sign of complex conjugation and sign of underline of the coordinates on the Lorentz group. Below, instead of using underlined and nonunderlined indices, we stipulate what kind of objects (particle or antiparticle) is described by the spin-tensor field under consideration.

As a simple example, we consider first linear in **z** functions, they correspond to spin $1/2$ particles. Suppose a particle is described by a function $f(x, z, \overset{*}{\underline{z}}) \in V_+$,

$$
\begin{aligned}
&f(x, z, \overset{*}{\underline{z}}) = \chi_\alpha(x) z^\alpha + \overset{*}{\psi}{}^{\dot\alpha}(x) \overset{*}{\underline{z}}_{\dot\alpha} = Z_D \Psi(x), \quad Z_D = (z^\alpha \overset{*}{\underline{z}}_{\dot\alpha}), \\
&\Psi(x) = \begin{pmatrix} \chi_\alpha(x) \\ \overset{*}{\psi}{}^{\dot\alpha}(x) \end{pmatrix},
\end{aligned} \tag{4.1}
$$

then an antiparticle is described by a function $f(x, \underline{z}, \overset{*}{z}) \in V_-$,

$$
f(x, \underline{z}, \overset{*}{z}) = \chi_\alpha(x) \underline{z}^\alpha + \overset{*}{\psi}{}^{\dot\alpha}(x) \overset{*}{z}_{\dot\alpha} = \underline{Z}_D \Psi(x), \quad \underline{Z}_D = (\underline{z}^\alpha \overset{*}{z}_{\dot\alpha}), \tag{4.2}
$$

where Z_D and $\underline{Z}_D$ (and therefore bispinor $\Psi(x)$ in both equations) have the same transformation law under the proper Poincaré group $M(3, 1)$. Using Eqs. (3.16) and (3.19), we get for space reflection

$$
P : Z_D \Psi(x) \to Z_D \Psi^{(s)}(\overline{x}), \quad \Psi^{(s)}(\overline{x}) = -\begin{pmatrix} \overset{*}{\psi}{}^{\dot\alpha}(\overline{x}) \\ \chi_\alpha(\overline{x}) \end{pmatrix} = \gamma^0 \Psi(\overline{x}).
$$

For the time reflection, we get $\Psi^{(s)}(\overline{x}) = \gamma^0 \Psi(-\overline{x})$. The charge conjugation corresponds to the complex conjugation in the space of the scalar functions. Thus, according to (3.20), we write

$$
C : Z_D \Psi(x) \to \overset{*}{Z}_D \overset{*}{\Psi}(x) = \underline{Z}_D \Psi^{(c)}(x), \quad \Psi^{(c)}(x) = -\begin{pmatrix} \psi_\alpha(x) \\ \overset{*}{\chi}{}^{\dot\alpha}(x) \end{pmatrix} = i\gamma^2 \overset{*}{\Psi}(x).
$$

Finally, using Eqs. (3.26) and (3.27), we obtain the action of the discrete transformations I_z and I_3,

$$
\begin{aligned}
I_z &: Z_D \Psi(x) \to \underline{Z}_D \gamma^5 \Psi(x), \\
I_3 &: Z_D \Psi(x) \to -Z_D i \Psi(x).
\end{aligned}
$$

Both transformations I_z and C interchange particles and antiparticles. The transformation I_3 is reduced to a multiplication by a phase factor only.

In order to find transformation laws for general spin-tensor fields we need an explicit form for bases of the Lorentz group irreps. Consider the monomial basis

$$
(z^1)^a (z^2)^b (\overset{*}{\underline{z}}_1)^c (\overset{*}{\underline{z}}_2)^d
$$

in the space of functions $\phi(z, \overset{*}{\underline{z}})$. The numbers $j_1 = (a+b)/2$ and $j_2 = (c+d)/2$ are not changed under the action of the Lorentz group generators (A1). Hence the space of the irrep (j_1, j_2) is the space of homogeneous functions of two pairs of complex variables of power $(2j_1, 2j_2)$. We denote these functions as $\varphi_{j_1 j_2}(z)$.

For finite-dimensional nonunitary irreps of $SL(2, C)$, the numbers a, b, c, d are integer nonnegative, therefore, j_1, j_2 are integer or half-integer nonnegative.

Consider polynomials $f_s(x,z)$ of degree $2s = 2j_1 + 2j_2$ in $z, \overset{*}{\underline{z}}$,

$$f_s(x,z,\overset{*}{\underline{z}}) = \sum_{j_1+j_2=s} \sum_{m_1,m_2} \psi^{m_1 m_2}_{j_1 j_2}(x)\varphi^{m_1 m_2}_{j_1 j_2}(z,\overset{*}{\underline{z}}). \tag{4.3}$$

The functions

$$\varphi^{m_1 m_2}_{j_1 j_2}(z,\overset{*}{\underline{z}}) = N^{\frac{1}{2}}(z^1)^{j_1+m_1}(z^2)^{j_1-m_2}(\overset{*}{\underline{z}}_{\dot{1}})^{j_2+m_2}(\overset{*}{\underline{z}}_{\dot{2}})^{j_2-m_2}, \tag{4.4}$$

$$N = (2s)![(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!]^{-1} \tag{4.5}$$

form a basis of an irrep of the Lorentz group. Such a basis corresponds to a chiral representation. On the other hand, we can write a decomposition for the same functions in terms of symmetric spin-tensors $\psi_{\alpha_1\ldots\alpha_{2j_1}}{}^{\dot{\beta}_1\ldots\dot{\beta}_{2j_2}}(x) = \psi_{\alpha_{(1}\ldots\alpha_{2j_1)}}{}^{\dot{\beta}_{(1}\ldots\dot{\beta}_{2j_2)}}(x)$. Namely,

$$\begin{aligned} f_s(x,z) &= \sum_{j_1+j_2=s} f_{j_1 j_2}(x,z), \\ f_{j_1 j_2}(x,z) &= \psi_{\alpha_1\ldots\alpha_{2j_1}}{}^{\dot{\beta}_1\ldots\dot{\beta}_{2j_2}}(x) z^{\alpha_1}\cdots z^{\alpha_{2j_1}} \overset{*}{\underline{z}}_{\dot{\beta}_1}\cdots\overset{*}{\underline{z}}_{\dot{\beta}_{2j_2}}. \end{aligned} \tag{4.6}$$

Comparing the decompositions (4.3) and (4.6), we obtain the relation

$$N^{\frac{1}{2}}\psi^{m_1 m_2}_{j_1 j_2}(x) = \psi \begin{matrix} \overbrace{\dot{1}\cdots\dot{1}}^{j_2+m_2}\ \overbrace{\dot{2}\cdots\dot{2}}^{j_2-m_2} \\ \underbrace{1\cdots 1}_{j_1+m_1}\ \underbrace{2\cdots 2}_{j_1-m_1} \end{matrix}(x). \tag{4.7}$$

Consider now the action of the discrete transformations on the functions $\psi(x)$. According to (3.16) and (3.19), the automorphism that is related to P allows us to write (see (4.3) and (4.4))

$$f(x,z,\overset{*}{\underline{z}}) \xrightarrow{P} f(\overline{x}, -\overset{*}{\underline{z}}, -z) = \varphi(-\overset{*}{\underline{z}}, -z)\psi(\overline{x}) = \varphi(z,\overset{*}{\underline{z}})\psi^{(s)}(\overline{x}). \tag{4.8}$$

It follows from (4.4) that

$$\varphi^{m_1 m_2}_{j_1 j_2}(-\overset{*}{\underline{z}}, -z) = (-1)^{2(j_1+j_2)}\varphi^{m_2 m_1}_{j_2 j_1}(z,\overset{*}{\underline{z}}), \tag{4.9}$$

therefore we get

$$\psi^{(s)m_1 m_2}_{j_1 j_2}(\overline{x}) = (-1)^{2(j_1+j_2)}\psi^{m_2 m_1}_{j_2 j_1}(\overline{x}).$$

Finally, we get P transformation of spin-tensor fields

$$\psi_{\alpha_1\cdots\alpha_{2j_1}}{}^{\dot{\beta}_1\cdots\dot{\beta}_{2j_2}}(x) \xrightarrow{P} (-1)^{2(j_1+j_2)}\psi_{\beta_1\cdots\beta_{2j_2}}{}^{\dot{\alpha}_1\cdots\dot{\alpha}_{2j_1}}(\overline{x}). \tag{4.10}$$

The charge conjugation C maps functions $f(x,z,\overset{*}{\underline{z}}) \in V_+$ into functions $f(x,\underline{z},\overset{*}{z}) \in V_-$:

$$f(x,z,\overset{*}{\underline{z}}) \xrightarrow{C} \overset{*}{f}(x,z,\overset{*}{\underline{z}}) = \overset{*}{\varphi}(z,\overset{*}{\underline{z}})\psi^*(x) = \varphi(\underline{z},\overset{*}{z})\psi^{(c)}(x). \tag{4.11}$$

Using again (4.4) to write

$$\overset{*}{\varphi}{}^{m_1 m_2}_{j_1 j_2}(z, \underline{\overset{*}{z}}) = \varphi^{m_1 m_2}_{j_1 j_2}(\overset{*}{z}, \underline{z}) = (-1)^{(j_1-m_1)+(j_2+m_2)}\varphi^{m_2 m_1}_{j_2 j_1}(\underline{z}, \overset{*}{z}), \tag{4.12}$$

we obtain

$$\psi^{(c) m_1 m_2}_{j_1 j_2}(x) = (-1)^{(j_1-m_1)+(j_2+m_2)}\overset{*}{\psi}{}^{m_2 m_1}_{j_2 j_1}(x).$$

Thus C transformation of spin-tensor fields has the form

$$\begin{aligned} \psi_{\alpha_1\cdots\alpha_{2j_1}}{}^{\dot\beta_1\cdots\dot\beta_{2j_2}}(x) &\overset{C}{\to} \overset{*}{\psi}{}^{\beta_1\cdots\beta_{2j_2}}{}_{\dot\alpha_1\cdots\dot\alpha_{2j_1}}(x) \\ &= (-1)^{(j_1-m_1)+(j_2+m_2)}\overset{*}{\psi}{}_{\beta_1\cdots\beta_{2j_2}}{}^{\dot\alpha_1\cdots\dot\alpha_{2j_1}}(x). \end{aligned} \tag{4.13}$$

The action of discrete transformations on functions $f(x, \underline{z}, \overset{*}{z}) \in V_-$, which correspond to antiparticle fields, can be obtained a similar manner. Below we summarize transformation laws both for scalar fields on the Poincaré group and for spin-tensors fields in Minkowski space in two tables.

Besides of the five independent transformations P, T, C, I_z, I_3, we include in these tables two operations related to the change of time sign (Wigner time reversal T_{w} and Schwinger time reversal T_{sch}), the inversion I_x (which affects only space-time coordinates x^μ), and PCT_{w}-transformation.

It is easy to see that $C^2 = P^2 = T^2 = 1$. Operators I_z, I_3 correspond to products of involutory inner automorphisms and the rotation by the angle π (see (3.24)). Hence $I_z^2 = I_3^2 = T_{\mathrm{w}}^2 = R_{2\pi}$, where $R_{2\pi}$ is the operator of rotation by 2π. The latter operation changes signs of spin variables, $f(x, z, \underline{\overset{*}{z}}) \overset{R_{2\pi}}{\to} f(x, -z, -\underline{\overset{*}{z}})$ and corresponds to the multiplication by the phase factor $(-1)^{2(j_1+j_2)}$ only.

In the general case the transformation laws for particle and antiparticle spin-tensor fields are distinguished by signs (for space reflection this fact was pointed out in the literature, see, for example, Sachs, 1987). This signs play an important role, because their change leads to noncommutativity of discrete transformations.

There are two different transformations C and I_z, which interchange particle and antiparticle fields. The operator I_z is a spin part of PCT_{w}-transformation. Indeed, the relation $PCT_{\mathrm{w}} = I_x I_z$ means that PCT_{w}-transformation is factorized in inversion I_x, affecting only space-time coordinates x^μ, and in I_z-transformation, affecting only spin coordiantes $\mathbf{z}$.

Consider now scalar fields that are eigenfunctions for C. Such fields describe neutral particles and obey the condition $Cf(h) = \overset{*}{f}(h) = e^{i\phi} f(h)$. Multiplying these fields by the phase factor $e^{i\phi/2}$, we transform them to real fields obeying the condition $Cf(h) = f(h)$. The charge conjugation C maps $z, \underline{z}$ into a complex conjugate pair. Thus, there are two invariant (with respect to C) subspaces of the scalar functions, namely, spaces of real-valued functions $f(x, z, \overset{*}{z})$ and $f(x, \underline{z}, \underline{\overset{*}{z}})$. We denote such spaces by V_z and $V_{\underline{z}}$, respectively. They are mapped into one another under the space reflection, $V_z \overset{P}{\leftrightarrow} V_{\underline{z}}$. Eigenfunctions of C that are linear in $z, \overset{*}{z}$ (with the eigenvalue 1) have the form

$$f(x, z, \overset{*}{z}) = \psi_\alpha(x) z^\alpha - \overset{*}{\psi}{}^{\dot\alpha}(x) \overset{*}{z}_{\dot\alpha} = Z_{\mathrm{M}} \Psi_{\mathrm{M}}(x),$$
$$Z_{\mathrm{M}} = (z^\alpha \overset{*}{z}_{\dot\alpha}), \quad \Psi_{\mathrm{M}}(x) = \begin{pmatrix} \psi_\alpha(x) \\ -\overset{*}{\psi}{}^{\dot\alpha}(x) \end{pmatrix}, \tag{4.14}$$

where $\Psi_{\mathrm{M}}(x)$ is a Majorana spinor, $\overset{*}{\Psi}_{\mathrm{M}}(x) = i\gamma^2 \Psi_{\mathrm{M}}(x)$. The space reflection maps functions from V_z into functions from $V_{\underline{z}}$,

$$P : Z_{\mathrm{M}} \Psi_{\mathrm{M}}(x) \to \underline{Z}_{\mathrm{M}} \Psi^{(s)}_{\mathrm{M}}(\overline{x}), \quad \underline{Z}_{\mathrm{M}}(\underline{z}^\alpha \overset{*}{\underline{z}}_{\dot\alpha}),$$
$$\Psi^{(s)}_{\mathrm{M}}(\overline{x}) = -\begin{pmatrix} \overset{*}{\psi}{}^{\dot\alpha}(\overline{x}) \\ \psi_\alpha(\overline{x}) \end{pmatrix} = \gamma^0\gamma^5 \Psi_{\mathrm{M}}(\overline{x}) = i\gamma^0\gamma^5\gamma^2 \overset{*}{\Psi}_{\mathrm{M}}(\overline{x}). \tag{4.15}$$

Therefore, the spaces V_z and $V_{\underline{z}}$ (in contrast to the spaces V_+ and V_-) do not contain eigenfunctions of P (i.e., states with definite parity). According to (3.26) and (3.27), we obtain

$$I_z : Z_{\mathrm{M}} \Psi(x) \to \underline{Z}_{\mathrm{M}} \Psi(x), \tag{4.16}$$

$$I_3 : Z_{\mathrm{M}} \Psi(x) \to -Z_{\mathrm{M}} i\gamma^5 \Psi(x). \tag{4.17}$$

Thus, there are four nontrivial independent discrete transformations for the fields under consideration. These transformations for bispinors $\Psi(x)$ and $\Psi_{\mathrm{M}}(x)$ are performed by matrices from the same set. However, one and the same discrete symmetry operation induces different operations with bispinors $\Psi(x)$ and $\Psi_{\mathrm{M}}(x)$.

The PCT_{w}-transformation maps the spaces of functions $f(x, z, \underline{z})$ and $f(x, \overset{*}{z}, \overset{*}{\underline{z}})$ into themselves. The eigenfunctions of PCT_{w} from these spaces describe, in particular, "physical" Majorana particles, which are defined as PCT_{w}-self-conjugate particles with spin $1/2$ (Kayser and Goldhaber, 1983).

5. DISCRETE TRANSFORMATIONS OF OPERATORS

First consider the action of the discrete transformations on generators of the Poincaré group. Such generators in the left generalized regular representation have the form

$$\hat{p}_\mu = -i\partial/\partial x^\mu, \quad \hat{J}_{\mu\nu} = \hat{L}_{\mu\nu} + \hat{S}_{\mu\nu}, \tag{5.1}$$

where $\hat{L}_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$ are orbital momentum operators and $\hat{S}_{\mu\nu}$ are spin operators. The latter operators depend on $\mathbf{z}$ and $\partial/\partial\mathbf{z}$, an explicit form of the operators is presented in the Appendix. The generators (5.1) obey the commutation relations

$$[\hat{p}_\mu, \hat{p}_\nu] = 0, \quad [\hat{J}_{\mu\nu}, \hat{p}_\rho] = i(\eta_{\nu\rho}\hat{p}_\mu - \eta_{\mu\rho}\hat{p}_\nu),$$
$$[\hat{J}_{\mu\nu}, \hat{J}_{\rho\sigma}] = i\eta_{\nu\rho}\hat{J}_{\mu\sigma} - i\eta_{\mu\rho}\hat{J}_{\nu\sigma} - i\eta_{\nu\sigma}\hat{J}_{\mu\rho} + i\eta_{\mu\sigma}\hat{J}_{\nu\rho}. \tag{5.2}$$

Fields on the Poincaré group depend on ten independent variables. For the classification of these fields we can use a maximal set of commuting operators. As such a set we chose both the left generators (5.1) and generators of the right generalized regular representation $T_{\mathrm{R}}(g)$,

$$T_{\mathrm{R}}(g) f(x,z) = f(xg, zg), \quad xg \leftrightarrow X + ZAZ^{\dagger}, \quad zg \leftrightarrow ZU. \tag{5.3}$$

Generators of the right generalized regular representation are labelled by R above. As a consequence of (5.3) we obtain

$$\hat{p}^{\mathrm{R}}_{\mu}\bar{\sigma}^{\mu} = -Z^{-1}\hat{p}_{\nu}\bar{\sigma}^{\nu}(Z^{\dagger})^{-1}, \quad \hat{J}^{\mathrm{R}}_{\mu\nu} = \hat{S}^{\mathrm{R}}_{\mu\nu}. \tag{5.4}$$

Operators $\hat{S}_{\mu\nu}$ and $\hat{S}^{\mathrm{R}}_{\mu\nu}$ from (5.1) and (5.4) are the left and right generators of $SL(2, C)$, they do not depend on x. All the right generators (5.4) commute with all the left generators (5.1) and obey the same commutation relations (5.2). Below for spin projection operators we use three-dimensional vector notation $\hat{S}_k = \frac{1}{2}\epsilon_{ijk}\hat{S}^{ij}$, $\hat{B}_k = \hat{S}_{0k}$. An explicit form of the spin operators is given in the Appendix by Eqs. (A1)–(A3).

According to harmonic analysis theory of Lie groups (Barut and Raczka, 1977; Zhelobenko and Schtern, 1983), a maximal set of commuting operators includes both Casimir operators and two sets of left and right generators (both sets in equal number). The total number of the commuting operators is equal to the number of group parameters. Nonequivalent representations (in a decomposition of the left GRR) are distinguished by eigenvalues of Casimir operators, equivalent representations are distinguished by eigenvalues of right generators, and states within an irrep are distinguished by eigenvalues of left generators.

In the general case, the physical meaning of right generators is not so transparent as of left ones. Nevertheless, right generators of $SO(3)$ in the nonrelativistic rotator theory are interpreted as angular momentum operators in a rotating body-fixed reference frame (Biedenharn and Louck, 1981; Landau and Lifschitz, 1977; Wigner, 1959). Since the right transformations commute with the left ones, they define quantum numbers, which do not depend on the choice of the laboratory reference frame.

The right generators $\hat{S}^{\mathrm{R}}_3$ and $\hat{B}^{\mathrm{R}}_3$ of the Poincaré group can be used to distinguish functions from the subspaces V_+, V_- and V_z, $V_{\underline{z}}$. Polynomials of power $2s$ belonging to V_+ and V_- are eigenfunctions of the operator $\hat{S}^{\mathrm{R}}_3$ with eigenvalues $\mp s$ respectively. Polynomials of power $2s$ belonging to V_z and $V_{\underline{z}}$ are eigenfunctions of the operator $i\hat{B}^{\mathrm{R}}_3$ with eigenvalues $\mp s$ respectively.

The explicit form of the generators (see (5.1), (5.4) and (A1), (A2)) allows us to find their transformation properties under involutory automorphisms, and, thus, under discrete transformations. The transformations P, T correspond to outer automorphisms of the algebra. Therefore, left and right generators are transformed similarly under P and T, in particular,

$$\hat{p}_\mu \to \mp(-1)^{\delta_{0\mu}}\hat{p}_\mu, \quad \hat{\mathbf{S}} \to \hat{\mathbf{S}}, \quad \hat{\mathbf{B}} \to -\hat{\mathbf{B}},$$

where the upper sign corresponds to P. Obviously, spatial and boost components of the total and orbital angular momenta are transformed as $\hat{\mathbf{S}}$ and $\hat{\mathbf{B}}$.

The complex conjugation C changes signs of all the generators. The corresponding commutation relations are also changed under the complex conjugation; they follow from (5.2) if to replace there i by $-i$.

According to (3.25)–(3.27) the transformations I_z, I_3, which are connected with inner automorphisms, are defined as right finite transformations of the proper Poincaré group.[9] They do not affect left generators since all the right transformations commute with all the left ones. Thus, I_z, I_3 induce automorphisms of the right generators algebra: I_z changes signs of the first and the third components of $\hat{\mathbf{S}}^{\mathrm{R}}$ and $\hat{\mathbf{B}}^{\mathrm{R}}$, and I_3 changes signs of the first and the second components of $\hat{\mathbf{S}}^{\mathrm{R}}$ and $\hat{\mathbf{B}}^{\mathrm{R}}$.

An intrinsic parity of a massive particle is defined as an eigenvalue of the operator P in the rest frame, $Pf(h) = \eta f(h)$, $\eta = \pm 1$. Since the operator P commutes with T, C, I_z, the intrinsic parity is not changed under the corresponding discrete transformations.

Using information from Tables I and II and explicit form of the corresponding operators presented in the Appendix, one can find transformation properties of physical quantities under discrete transformations. Such properties are listed in Table III. The intrinsic parity η and the sign of p_0 label irreps of the improper Poincaré group. The latter group includes the proper Poincaré group and space reflection. As well as Table III includes the left generators $\hat{p}_\mu$, the spin parts $\hat{\mathbf{S}}$, $\hat{\mathbf{B}}$ of the left Lorentz generators, and two right Lorentz generators.

We also include in the table a current four-vector j_μ for the first-order equation (B2) (the Dirac and Duffin–Kemmer equations are the particular cases of this equation for $s = 1/2$ and $s = 1$ respectively). In the space of scalar functions on the group this current is presented by the opertors $\hat{\Gamma}_\mu$, see (B3). Particle and antiparticle fields are distinguished by the sign of the charge, that is, by the sign of j_0 component of the current. As one can see from the table, the sign of the eigenvalue S_3^{R} of right generator can be used to distinguish particles and antiparticles, since this sign and the sign of j_0 are transformed similarly under discrete transformations. As is shown below, the sign of the mass term in the Eq. (B2) is changed as the sign of the product $p_0 S_3^{\mathrm{R}}$ under discrete transformations, see the next to last column of Table III.

[9] Any inner automorphism $h \to g^{-1}hg$ is a product of left $h \to g^{-1}h$ and right $h \to hg$ group transformations. In other words, any inner automorphism of the proper Poincaré group can be reduced to a right transformation of this group by means of a corresponding choice of the reference frame. That gives us one more reason to study the right transformations.

Table I. Discrete Transformations for Particle Fields

	$f(x, z, \underline{\overset{*}{z}})$	$\psi_{\alpha_1\cdots\alpha_{2j_1}}{}^{\dot\beta_1\cdots\dot\beta_{2j_2}}(x)$	$\Psi(x)$
P	$f(\bar{x}, -\underline{\overset{*}{z}}, -z)$	$(-1)^{2(j_1+j_2)}\psi_{\beta_1\cdots\beta_{2j_2}}{}^{\dot\alpha_1\cdots\dot\alpha_{2j_1}}(\bar{x})$	$\gamma^0\Psi(\bar{x})$
T	$f(-\bar{x}, -\underline{\overset{*}{z}}, -z)$	$(-1)^{2(j_1+j_2)}\psi_{\beta_1\cdots\beta_{2j_2}}{}^{\dot\alpha_1\cdots\dot\alpha_{2j_1}}(-\bar{x})$	$\gamma^0\Psi(-\bar{x})$
$I_x = PT$	$f(-x, z, \underline{\overset{*}{z}})$	$\psi_{\alpha_1\cdots\alpha_{2j_1}}{}^{\dot\beta_1\cdots\dot\beta_{2j_2}}(-x)$	$\Psi(-x)$
C	$\overset{*}{f}(x, z, \underline{\overset{*}{z}})$	$\overset{*}{\psi}{}^{\beta_1\cdots\beta_{2j_2}}{}_{\dot\alpha_1\cdots\dot\alpha_{2j_1}}(x)$	$i\gamma^2\overset{*}{\Psi}(x)$
$T_{\text{sch}} = CT$	$\overset{*}{f}(-\bar{x}, -\underline{\overset{*}{z}}, -z)$	$(-1)^{2(j_1+j_2)}\overset{*}{\psi}{}^{\alpha_1\cdots\alpha_{2j_1}}{}_{\dot\beta_1\cdots\dot\beta_{2j_2}}(-\bar{x})$	$i\gamma^0\gamma^2\overset{*}{\Psi}(-\bar{x})$
I_z	$f(x, \underline{z}, -\overset{*}{z})$	$(-1)^{2j_2}\psi_{\alpha_1\cdots\alpha_{2j_1}}{}^{\dot\beta_1\cdots\dot\beta_{2j_2}}(x)$	$\gamma^5\Psi(x)$
$T_{\text{w}} = I_zCT$	$\overset{*}{f}(-\bar{x}, \overset{*}{z}, -\underline{z})$	$(-1)^{2j_2}\overset{*}{\psi}{}^{\alpha_1\cdots\alpha_{2j_1}}{}_{\dot\beta_1\cdots\dot\beta_{2j_2}}(-\bar{x})$	$-i\gamma^5\gamma^0\gamma^2\overset{*}{\Psi}(-\bar{x})$
$PCT_{\text{w}} = I_zI_x$	$f(-x, \underline{z}, -\overset{*}{z})$	$(-1)^{2j_2}\psi_{\alpha_1\cdots\alpha_{2j_1}}{}^{\dot\beta_1\cdots\dot\beta_{2j_2}}(-x)$	$\gamma^5\Psi(-x)$
I_3	$f(x, -iz, -i\underline{\overset{*}{z}})$	$(-i)^{2(j_1+j_2)}\psi_{\alpha_1\cdots\alpha_{2j_1}}{}^{\dot\beta_1\cdots\dot\beta_{2j_2}}(x)$	$-i\Psi(x)$

The last column (L–R) of the table describes the passage between two types of spinors (left $\overset{*}{z}_{\dot\alpha}$, $\underline{\overset{*}{z}}_{\dot\alpha}$ and right z^α, $\underline{z}^\alpha$) labelled by dotted and undotted indices. The sign $-$ corresponds to a transformation interchanging dotted and undotted indices; the sign $+$ corrresponds to a transformation that does not change this indices. Let we define the chirality as a difference between the number of dotted and undotted indices, then the last column of the table corresponds to the sign of the chirality. In

Table II. Discrete Transformations for Antiparticle Fields

	$f(x, \underline{z}, \overset{*}{z})$	$\psi_{\alpha_1\cdots\alpha_{2j_1}}{}^{\dot\beta_1\cdots\dot\beta_{2j_2}}(x)$	$\Psi(x)$
P	$f(\bar{x}, \overset{*}{z}, \underline{z})$	$\psi_{\beta_1\cdots\beta_{2j_2}}{}^{\dot\alpha_1\cdots\dot\alpha_{2j_1}}(\bar{x})$	$-\gamma^0\Psi(\bar{x})$
T	$f(\bar{x}, \overset{*}{z}, \underline{z})$	$\psi_{\beta_1\cdots\beta_{2j_2}}{}^{\dot\alpha_1\cdots\dot\alpha_{2j_1}}(-\bar{x})$	$-\gamma^0\Psi(-\bar{x})$
$I_x = PT$	$f(-x, \underline{z}, \overset{*}{z})$	$\psi_{\alpha_1\cdots\alpha_{2j_1}}{}^{\dot\beta_1\cdots\dot\beta_{2j_2}}(-x)$	$\Psi(-x)$
C	$\overset{*}{f}(x, z, \underline{\overset{*}{z}})$	$\overset{*}{\psi}{}^{\beta_1\cdots\beta_{2j_2}}{}_{\dot\alpha_1\cdots\dot\alpha_{2j_1}}(x)$	$i\gamma^2\overset{*}{\Psi}(x)$
$T_{\text{sch}} = CT$	$\overset{*}{f}(-\bar{x}, \overset{*}{z}, \underline{z})$	$\overset{*}{\psi}{}^{\alpha_1\cdots\alpha_{2j_1}}{}_{\dot\beta_1\cdots\dot\beta_{2j_2}}(-\bar{x})$	$-i\gamma^0\gamma^2\overset{*}{\Psi}(-\bar{x})$
I_z	$f(x, -z, \underline{\overset{*}{z}})$	$(-1)^{2j_1}\psi_{\alpha_1\cdots\alpha_{2j_1}}{}^{\dot\beta_1\cdots\dot\beta_{2j_2}}(x)$	$-\gamma^5\Psi(x)$
$T_{\text{w}} = I_zCT$	$\overset{*}{f}(-\bar{x}, \underline{\overset{*}{z}}, -z)$	$(-1)^{2j_2}\overset{*}{\psi}{}^{\alpha_1\cdots\alpha_{2j_1}}{}_{\dot\beta_1\cdots\dot\beta_{2j_2}}(-\bar{x})$	$-i\gamma^5\gamma^0\gamma^2\overset{*}{\Psi}(-\bar{x})$
$PCT_{\text{w}} = I_zI_x$	$f(-x, -z, \underline{\overset{*}{z}})$	$(-1)^{2j_1}\psi_{\alpha_1\cdots\alpha_{2j_1}}{}^{\dot\beta_1\cdots\dot\beta_{2j_2}}(-x)$	$-\gamma^5\Psi(-x)$
I_3	$f(x, i\underline{z}, i\overset{*}{z})$	$-i^{2(j_1+j_2)}\psi_{\alpha_1\cdots\alpha_{2j_1}}{}^{\dot\beta_1\cdots\dot\beta_{2j_2}}(x)$	$i\Psi(x)$

Table III. Discrete Transformation Action on Signs of Physical Quantities

	η	p_0	$\mathbf{p}$	$\mathbf{S}$	$\mathbf{B}$	S_3^{R}	B_3^{R}	j_0^a	$\boldsymbol{j}^a$	$p_0 S_3^{\mathrm{R}}$	L–R
P	+	+	−	+	−	+	−	+	−	+	−
T	+	−	+	+	−	+	−	+	−	−	−
$I_x = PT$	+	−	−	+	+	+	+	+	+	−	+
C	+	−	−	−	−	−	−	−	−	+	−
CP	+	−	+	−	+	−	+	−	+	+	+
$T_{\mathrm{sch}} = CT$	+	+	−	−	+	−	+	−	+	−	+
$PCT = I_x C$	+	+	+	−	−	−	−	−	−	−	−
I_z	+	+	+	+	+	−	−	−	−	−	+
$T_{\mathrm{w}} = I_z CT$	+	+	−	−	+	+	−	+	−	+	+
$PT_{\mathrm{w}} = I_z I_x C$	+	+	+	−	−	+	+	+	+	+	−
$PCT_{\mathrm{w}} = I_z I_x$	+	−	−	+	+	−	−	−	−	+	+

[a] For the states described by the first-order equation (B2).

the space of scalar functions on the group, the chirality is described by the operator $\hat{\Gamma}^5$, see (B5).

The time reflection T maps positive energy states into negative energy ones. On the other hand, the time reversal is defined usually by the relation $x \to -\bar{x}$ with supplementary condition of conservation of the energy sign. Obviously, the product of charge conjugation and time reflection CT, which we denote by T_{sch}, obeys this condition. The transformation T_{sch} was introduced by Schwinger (1951) (see also Umezava *et al.*, 1954). This transformation interchanges particle and antiparticle fields (fields with opposite signs of j_0 component).

For the first time, the time reversal T_{w} was considered by Wigner (1932). Wigner time reversal does not change the sign of j_0. Relating different states of the same particle, this transformation is an analog of the time reversal in nonrelativistic quantum mechanics. Changing signs of the vectors $\mathbf{p}$, $\mathbf{S}$, $\mathbf{j}$, Wigner time reversal corresponds to a reversal of the motion direction. Notice that sometimes the term "time reversal" is used (instead of the term "time reflection") for transformations changing the sign of energy.

The transformation I_z is a finite transformation from the right generalized regular representation of the proper Poincaré group, see (3.25) and (3.27). This transformation does not change signs of the left generators (since all the right transformations commute with the left ones) but changes signs of the current vector and of some right generators. Hence, the left generators are transformed similarly under T_{sch} and $T_{\mathrm{w}} = I_z T_{\mathrm{sch}}$. The transformation I_3 (as it was mentioned above) changes the sign of the first and second components of the vectors $\mathbf{S}^{\mathrm{R}}$ and $\mathbf{B}^{\mathrm{R}}$ and does not change signs of all the physical quantities listed in the Table III.

6. EXTENSION OF THE PROPER POINCARÉ GROUP IN REPRESENTATION SPACES BY DISCRETE TRANSFORMATIONS

Since we have a clear definition of discrete transformations in representation spaces (or equivalently in the space of scalar functions on the group), we can consider an extension of the proper Poincaré group by means of the discrete transformations (further we call such a group the extended Poincaré group). Irreps of the extended Poincaré group can differ from ones of the proper Poincaré group, because discrete transformations can unite nonequivalent irreps or distinguish equivalent irreps of the proper Poincaré group. Indeed, different fields with identical transformation rules under the left transformations carry equivalent subrepresentations of the left generalized regular representation (3.1) even if these fields have different transformation rules under the right transformations (5.3). The functions carrying equivalent representations of the proper Poincaré group can be transformed differently under the discrete transformations. Therefore, these functions carry nonequivalent representations of the extended Poincaré group.

The reflection I and the identity operator form a finite group Z_2, which consists of two elements. The operator I distinguishes states having different "charges" and "charge parities." States with opposite "charges" (which we denote by ψ_+ and ψ_-) are interchanged by the action of I: $\psi_+ \overset{I}{\leftrightarrow} \psi_-$. The states $\psi_+ \pm \psi_-$ with definite "charge parity" are eigenfunctions of I with the eigenvalues ± 1 respectively. These states form a basis of one-dimensional irreps of Z_2. The operators $(1 \pm I)/2$ are projection operators on states with definite "charge parity."

Operators of discrete transformations commute between each other and commute (sign "+" in Table III) or anticommute (sign "−" in Table III) with generators of the proper Poincaré group. The latter means that discrete transformation can only interchange eigenfunctions of the generator with opposite eigenvalues.

Below one can find a table, which lists parameters labelling finite-component (with respect to spin) irreps of the proper and improper (i.e., extended by the space reflection) Poincaré groups.

Here the mass $m > 0$, the spin $s = 0, 1/2, 1, \ldots$, the intrinsic parity $\eta = \pm 1$, and the helicity $\lambda = 0, \pm 1/2, \pm 1, \ldots$. The mass m and sign of p_0 label orbits in the momentum space (the upper or lower sheet of hyperboloid or cone), s and λ label irreps of the little groups $SO(3)$ and $SO(2)$, s, η and $|\lambda|$, η label irreps of the little groups $O(3)$ and $O(2)$, respectively (see Mackey, 1968; Tung, 1985, for details). The mass and the spin can by also defined as eigenvalues of Casimir operators:

$$\hat{p}^2 f(x, \mathbf{z}) = m^2 f(x, \mathbf{z}), \quad \hat{W}^2 f(x, \mathbf{z}) = -m^2 s(s+1) f(x, \mathbf{z}), \qquad (6.1)$$

where $\hat{W}^\mu$ is Lubanski–Pauli four-vector, and $\mathbf{z} = (z, \underline{z}, \overset{*}{z}, \overset{*}{\underline{z}})$ are coordinates on the Lorentz group.

To construct irreps of the extended Poincaré group, we consider the action of four independent discrete transformations P, I_x, I_z, C in the space of the scalar functions on the group.

1. Irreps of the extended by space reflection P Poincaré group (improper Poincaré group) can be classified with the help of the little group method. The space reflection distinguishes states with different intrinsic parties η and states with different (left or right) charges, or states with different chirality. In the space of scalar functions, the chirality operator $\hat{\Gamma}^5$ is given by Eq. (B5). Fields with zero chirality do not change under the space reflection.
2. The inversion I_x affects space-time coordinates x only. It couples two irreps of the proper (or improper) Poincaré group characterized by sign $p_0 = \pm 1$ into one representation of the extended group. Eigenvectors of I_x are states with definite "energy parity." However, since the sign of energy p_0 is already used to label irreps of the proper group, this extension does not creates new characteristics.
3. As was mentioned earlier, the operator I_z is a spin part of the $PCT_{\mathrm{w}} = I_x I_z$ transformation and affects only spin coordinates $\mathbf{z}$. Since I_z commutes with all the left generators and with space reflection P, I_z cannot change parameters labelling irreps of the proper or improper Poincaré group. I_z interchanges states with opposite eigenvalues of $\hat{S}_3^{\mathrm{R}}$; a charge parity $\eta_{\mathrm{c}} = \pm 1$ arises as eigenvalue of I_z. Therefore, irreps of the extended by I_z Poincaré group are labelled by charge parity η_{c} in addition to characteristics of Table IV. Furhter, taking into account the close relation between I_z and PCT_{w}, we call the corresponding characteristics as PCT_{w}-charge and PCT_{w}-parity.
4. The charge conjugation C changes signs of all the generators. Thus, any extension that includes C has to be considered separately. Notice that C does not change η and η_{c} and similar to I_z changes sign of the charge $\hat{S}_3^{\mathrm{R}}$. Whenever the proper Poincaré group is extended by P, T, I_z, then additional extension by C can be replaced by Wigner time reversal $T_{\mathrm{w}} = I_z C T$ as fourth independent discrete transformation. The latter transformation

Table IV. Parameters Labelling Irreps of the Proper and Improper Poincaré Groups

	Proper Poincaré group	Improper Poincaré group
Massive case	m, sign p_0, s	m, sign p_0, s, η
Massless case	sign p_0, λ	sign p_0, $\lvert\lambda\rvert$, η^a

[a]For $\lambda \neq 0$ massless irreps with $y = \pm 1$ are equivalent (Shaw and Lever, 1974; Tung, 1985).

corresponds to a reversal of particle motion direction and does not change P-charge (chirality) and $I_z(CPT_{\rm w})$-charge.

Thus, the irreps of the extended Poincaré group have two supplementary characteristics with respect to irreps of the proper group: the intrinsic parity η and $PCT_{\rm w}$-parity $\eta_{\rm c}$, which are associated with P-charge (chirality) and $PCT_{\rm w}$-charge (the latter one distinguishes particles and antiparticles). In the space of functions on the group these charges are defined as eigenvalues of the operators $\hat{\Gamma}^5$ and $\hat{S}_3^{\rm R}$, respectively. Since for particles with half-integer spins these charges are half-integer, such particles can't have zero chirality or zero $PCT_{\rm w}$-charge (in other words, they can't be pure neutral with respect to the discrete transformations under consideration).

Below we consider two types of extended Poincaré group representations. Irreps of the extended Poincaré group have definite intrinsic parity η and $PCT_{\rm w}$-charge parity $\eta_{\rm c}$. Fields with definite intrinsic parity η or with definite charge parity $\eta_{\rm c}$ (e.g., "physical" Majorana field) are described by eigenfunctions of P or I_z, respectively. Representations with definite P-charge (chirality) or with definite $PCT_{\rm w}$-charge are reducible representations of the extended Poincaré group. Fields with definite P-charge (e.g., Weyl field) or with definite $PCT_{\rm w}$-charge (e.g., Dirac field) are mapped into fields with opposite charges under the corresponding discrete transformations.

7. DISCRETE SYMMETRIES OF RELATIVISTIC WAVE EQUATIONS. MASSIVE CASE

Here we explicitly construct massive fields on the Poincaré group and analyze their characteristics associated with the discrete transformations. On this base, using various sets of commuting operators on the Poincaré group, we give a compact group-theoretical derivation of basic relativistic wave equations and consider their discrete symmetries. In particular, this allows one to present a group-theoretical interpretation of two possible signs of mass term in first order equations. Then we classify solutions of higher spin relativistic wave equations with respect to the extended Poincaré group.

Consider eigenfunctions of the operators $\hat{p}_\mu$ (plane waves). For $m \neq 0$, there exists a rest frame, where x-dependence is reduced to the factor $e^{\pm imx^0}$. Linear in $\mathbf{z}$ functions describe spin $1/2$ particles. For a fixed mass m, there are 16 linearly independent functions of such kind,

$$\begin{array}{ccc} & V_+ & V_- \\ L: & e^{\pm imx^0} z^\alpha & e^{\pm imx^0} \underline{z}^\alpha \\ R: & e^{\pm imx^0} \overset{*}{\underline{z}}_{\dot\alpha} & e^{\pm imx^0} \overset{*}{z}_{\dot\alpha} \end{array} \tag{7.1}$$

These functions can be classified (labelled) by means of left generators of the Poincaré group and operators of the discrete transformations P, C. The eigenvalues of the left generators $\hat{J}_3$ (in the rest frame under consideration $\hat{J}_3 = \hat{S}_3$) and $\hat{p}^0$ are the spin projection (for $\alpha, \dot{\alpha} = 1$ and $\alpha, \dot{\alpha} = 2$, we have $s_3 = 1/2$ respectively) and the energy p_0. The sign of p_0, along with the mass m and the spin s, characterizes nonequivalent irreps of the proper Poincaré group. The operator I_x interchanges states with opposite signs of p_0; the operator P interchanges L- and R-states (states with opposite chiralities); and the operators C and I_z interchange particle and antiparticle states. The latter states belong to the spaces V_+ and V_- respectively. In contrast to the operator C, the operator I_z does not change signs of energy and chirality.

However, the states (7.1) with definite chirality are transformed under reducible representation of the improper Poincaré group. Irreps of the latter group are characterized by the intrinsic parity η. In the rest frame, states with definite η are eigenfunctions of the operator $P, Pf(x, \mathbf{z}) = \eta f(x, \mathbf{z})$,

$$\begin{array}{rll} & V_+ & V_- \\ \eta = -1: & e^{\pm imx^0}(z^\alpha + \overset{*}{\underline{z}}_{\dot\alpha}) & e^{\pm imx^0}(\underline{z}^\alpha - \overset{*}{z}_{\dot\alpha}) \\ \eta = 1: & e^{\pm imx^0}(z^\alpha - \overset{*}{\underline{z}}_{\dot\alpha}) & e^{\pm imx^0}(\underline{z}^\alpha + \overset{*}{z}_{\dot\alpha}) \end{array} \tag{7.2}$$

As above, the operators C and I_z interchange functions from the spaces V_+ and V_-. On the other hand, states with different intrinsic parity $\eta = \pm 1$ (unlike states with different chirality) are not interchanged by operators of discrete transformations.

Both the states (7.1) and (7.2) are eigenvectors of the Casimir operators $\hat{p}^2$ and $\hat{W}^2$ with the eigenvalues m^2 and $-(3/4)m^2$. But the only states (7.2) are transformed under irrep of the improper Poincaré group. Besides, the states (7.2) (unlike the states (7.1)) are solutions of the equations

$$(\hat{p}_\mu \hat{\Gamma}^\mu \pm ms) f(x, z, \overset{*}{\underline{z}}) = 0, \quad (\hat{p}_\mu \hat{\Gamma}^\mu \mp ms) f(x, \underline{z}, \overset{*}{z}) = 0. \tag{7.3}$$

Here $s = 1/2$, the upper sign corresponds to η sign $p_0 = 1$ and lower sign corresponds to η sign $p_0 = -1$. The operator $\hat{p}_\mu \hat{\Gamma}^\mu$ (an explicit form of $\hat{\Gamma}^\mu$ is given by (B3)) is not affected by the space inversion and the charge conjugation. The spaces V_+ and V_- are also invariant under the space reflection, however, they are interchanged under the charge conjugation.

Considering the action of the discrete transformations on j_0 component of free equation current, we have seen that whenever particles are described by functions from V_+, then antiparticles are described by functions from V_-. This turns especially clear when we include an interaction with an external electromagnetic field. Acting by the charge conjugation C (which acts as the operator of complex conjugation on scalar functions on the group) on the equation

$$((\hat{p}_\mu - eA_\mu(x))\hat{\Gamma}^\mu \pm ms) f(x, z, \underline{\overset{*}{z}}) = 0, \quad f(x, z, \underline{\overset{*}{z}}) \in V_+, \tag{7.4}$$

we see that functions $\overset{*}{f}(x, z, \underline{\overset{*}{z}}) \in V_-$ obey the same equation with opposite charge sign:

$$((\hat{p}_\mu + eA_\mu(x))\hat{\Gamma}^\mu \pm ms) \overset{*}{f}(x, z, \underline{\overset{*}{z}}) = 0, \quad \overset{*}{f}(x, z, \underline{\overset{*}{z}}) \in V_-. \tag{7.5}$$

Substituting the functions $f(x, z, \underline{\overset{*}{z}}) = Z_D \Psi(x)$ and $\overset{*}{f}(x, \underline{z}, \overset{*}{z}) = \underline{Z}_D \Psi^{(c)}(x)$ into Eqs. (7.4) and (7.5) (see also (4.1) and (4.2)), we obtain two Dirac equations for charge-conjugate bispinors $\Psi(x)$ and $\Psi^{(c)}(x)$,

$$((\hat{p}_\mu - eA_\mu(x))\gamma^\mu \pm m)\Psi(x) = 0, \quad ((\hat{p}_\mu + eA_\mu(x))\gamma^\mu \pm m)\Psi^{(c)}(x) = 0.$$

Thus we have to use the different scalar functions on the group to describe particles and antiparticles and hence two Dirac equations for both signs of charge, respectively. That matches completely with results of the consideration by Gavrilov and Gitman (2000), where it was shown that in the course of the consistent quantization of a classical model of spinning particle namely such (charge symmetric) quantum mechanics appears. Such a mechanics is completely equivalent to one-particle sector of the corresponding quantum field theory.

In the next section we continue to consider spin-1/2 case and give an exact group-theoretical formulation of conditions that lead to the Dirac equation.

In the general case for the classification of functions corresponding to higher spins one has to use a maximal set of the commuting operators on the group, for example,

$$\hat{p}_\mu, \ \hat{W}^2, \ \hat{\mathbf{p}}\hat{\mathbf{S}}, \ \hat{\mathbf{S}}^2 - \hat{\mathbf{B}}^2, \ \hat{\mathbf{S}}\hat{\mathbf{B}}, \hat{S}_3^{\mathrm{R}}, \hat{B}_3^{\mathrm{R}}. \tag{7.6}$$

This set includes functions of left and right generators. In the rest frame $\hat{\mathbf{p}}\hat{\mathbf{S}} = 0$, thus the maximal set can be obtained from (7.6) by changing $\hat{\mathbf{p}}\hat{\mathbf{S}}$ to $\hat{S}_3$. Functions from the spaces V_+ and V_- depend on eight real parameters, therefore, one can consider only eight operators. As such operators we chose (the problem of constructing maximal sets of commuting operators in representation spaces of the Poincaré group was discussed by Barut and Raczka (1977), Gitman and Shelepin (2001), and Hai (1969))

$$\hat{p}_\mu, \hat{W}^2, \hat{\mathbf{p}}\hat{\mathbf{S}} \ (\hat{S}_3 \text{ in the rest frame}), \ \hat{p}_\mu\hat{\Gamma}^\mu, \hat{S}_3^{\mathrm{R}}. \tag{7.7}$$

Consider eigenvalue problem for the operators (7.7). For functions from the spaces V_+ and V_- one can show that if an eigenvalue of $\hat{p}_\mu\hat{\Gamma}^\mu$ is equal to $\pm ms$, where $2s$ is the power of polynomial, then the eigenvalue of the operator $\hat{W}^2$ is also fixed and corresponds to the spin s (Gitman and Shelepin, 2001).

Thus, the equations

$$\hat{p}^2 f(x, \mathbf{z}) = m^2 f(x, \mathbf{z}), \quad \hat{p}_\mu \hat{\Gamma}^\mu f(x, \mathbf{z}) = \pm m s f(x, \mathbf{z}),$$
$$\hat{S}_3^{\mathrm{R}} f(x, \mathbf{z}) = \pm s f(x, \mathbf{z}), \tag{7.8}$$

pick out states with definite mass and spin.

Depending on the choice of the functional space and of the sign of the mass term, the second equation (7.8) can be written in one of the four forms:

$$\begin{array}{cc} V_+ & V_- \end{array}$$
$$(\hat{p}_\mu \hat{\Gamma}^\mu + ms) f(x, z, \overset{*}{\underline{z}}) = 0, \quad (\hat{p}_\mu \hat{\Gamma}^\mu - ms) f(x, \underline{z}, \overset{*}{z}) = 0, \tag{7.9}$$
$$(\hat{p}_\mu \hat{\Gamma}^\mu - ms) f(x, z, \overset{*}{\underline{z}}) = 0, \quad (\hat{p}_\mu \hat{\Gamma}^\mu + ms) f(x, \underline{z}, \overset{*}{z}) = 0. \tag{7.10}$$

In the rest frame and for definite m and s, solutions of Eqs. (7.9) and (7.10) are given by (7.11) and (7.12), respectively,

$$\begin{array}{cc} V_+ & V_- \end{array}$$
$$e^{\pm imx^0}(z^1 \pm \overset{*}{\underline{z}}_{\dot{1}})^{s+s_3}(z^2 \pm \overset{*}{\underline{z}}_{\dot{2}})^{s-s_3} \quad e^{\mp imx^0}(\underline{z}^1 \pm \overset{*}{z}_{\dot{1}})^{s+s_3}(\underline{z}^2 \pm \overset{*}{z}_{\dot{2}})^{s-s_3},$$
$$\eta \operatorname{sign} p_0 = -1, \tag{7.11}$$
$$e^{\mp imx^0}(z^1 \pm \overset{*}{\underline{z}}_{\dot{1}})^{s+s_3}(z^2 \pm \overset{*}{\underline{z}}_{\dot{2}})^{s-s_3} \quad e^{\pm imx^0}(\underline{z}^1 \pm \overset{*}{z}_{\dot{1}})^{s+s_3}(\underline{z}^2 \pm \overset{*}{z}_{\dot{2}})^{s-s_3},$$
$$\eta \operatorname{sign} p_0 = 1, \tag{7.12}$$

Here the sign of ηp_0 is specified for half-integer spins; for integer spins, all the solutions have $\eta = 1$. Solutions (7.11) and (7.12) are eigenfunctions for the Casimir operators $\hat{p}^2$, $\hat{W}^2$, and for spin projection operator $\hat{S}_3$ with the eigenvalues m^2, $-s(s+1)m^2$, and s_3, $-s \leq s_3 \leq s$ respectively.

For half-integer spins, a general solution of the system (7.8) with definite sign of the mass term has definite sign of ηp_0. Such a solution carries a reducible representation of the improper Poincaré group. The representation is a direct sum of two irreps with opposite signs of η and p_0. Hence, the general solution contains $2(2s+1)$ independent components. Since η is invariant under discrete transformations, the representation carried by the solution remains reducible with respect to the extended Poincaré group.

Thus, for half-integer spins, the sign of the mass term in the Eqs. (7.9) and (7.10) coincides with the sign of the product

$$\eta p_0 S_3^{\mathrm{R}}. \tag{7.13}$$

Recall that the sign of S_3^{R} distinguishes particles and antiparticles; this sign is fixed by the choice of the space V_+ or V_-. In each the space V_+ or V_-, the general solution carries a direct sum of two irreps of the improper Poincaré group, which are characterized by $\operatorname{sign} p_0$, η or $-\operatorname{sign} p_0$, $-\eta$. For integer spins, in each space V_+ or V_-, the general solution carries a direct sum of two irreps, which are characterized by fixed intrinsic parity $\eta = 1$ and different signs of p_0.

As was demonstrated above, the set (7.7), which includes the first order in $\partial/\partial x$ operator $\hat{p}_\mu \hat{\Gamma}^\mu$, specifies some characteristics of representations of the extended Poincaré group. As it was shown by Gitman and Shelepin (in press), in the case of finite-dimensional representations of the Lorentz group, the system (7.8) is equivalent to Bargmann–Wigner equations. In turn, for half-integer spins, the latter equations are equivalent to Rarita–Schwinger equations (Ohnuki, 1988). Hence, the structure of solutions of Bargmann–Wigner and Rarita–Schwinger equations is similar to the structure of solutions of Eq. (7.8).

Considering equations that fix not only m and s but some additional characteristics of the extended Poincaré group representations, such as energy or charge signs, we cannot be sure that all the discrete transformations are symmetry ones for such equations. For example, the discrete symmetry group for the Eqs. (7.9) and (7.10) with definite sing of the mass term (and therefore discrete symmetry groups of the Dirac and Duffin–Kemmer equations) includes the only transformations that do not change the sign of $p_0 S_3^{\mathrm{R}}$.

The transformations P, C, T_{w} do not change sign of $p_0 S_3^{\mathrm{R}}$ and, therefore, do not change sign of the mass term in the first order equations under consideration. An additional (fourth) independent transformation changes sing of $p_0 S_3^{\mathrm{R}}$ and correspondingly sign of the mass term. As such a transformation, we can consider, for example, the inversion I_x or Schwinger time reversal T_{sch}.

Majorana equations (associated with infinite-dimensional irreps of $SL(2, C)$, Majorana, 1932; Stoyanov and Todorov, 1968) are only invariant under discrete transformations that do not change sign of p_0 (Naka and Gotō, 1971; Oksak and Todorov, 1968).

On the other hand, there exists a formulation that admits all four independent discrete transformations as symmetry transformations. This formulation is based on the use of set (7.6) of commuting operators and the representations $(s0) \oplus (0s)$ of the Lorentz group. To fix a representation $(s0) \oplus (0s)$, one can use Casimir operators of the Lorentz group or (for the subspaces $V_\pm$) the operators $\hat{B}_3^{\mathrm{R}}$, $\hat{S}_3^{\mathrm{R}}$; the set (7.6) contains all these operators. It was shown (Gitman and Shelepin, 2001) that equations

$$\hat{p}^2 f(x, \mathbf{z}) = m^2 f(x, \mathbf{z}), \quad \hat{S}_3^{\mathrm{R}} f(x, \mathbf{z}) = \pm s f(x, \mathbf{z}), \quad i \hat{B}_3^{\mathrm{R}} f(x, \mathbf{z}) = \pm s f(x, \mathbf{z}) \tag{7.14}$$

fix the spin of scalar functions from $V_\pm$. In the rest frame solutions of Eq. (7.14) with a definite spin projection s_3 have the form (in contrast to (7.11) and (7.12), signs in exponents and in brackets can be chosen independently):

$$V_+ : e^{\pm imx^0}((z^1)^{s+s_3}(z^2)^{s-s_3} \pm (\overset{*}{\underline{z}}_{\dot{1}})^{s+s_3}(\overset{*}{\underline{z}}_{\dot{2}})^{s-s_3}), \tag{7.15}$$

$$V_- : e^{\pm imx^0}((\underline{z}^1)^{s+s_3}(\underline{z}^2)^{s-s_3} \pm (\overset{*}{z}_{\dot{1}})^{s+s_3}(\overset{*}{z}_{\dot{2}})^{s-s_3}). \tag{7.16}$$

The sign in brackets defines the intrinsic parity. For half-integer spins, the upper sign corresponds to $\eta = 1$ and the lower sign corresponds to $\eta = -1$. For integer spins, the upper sign in (7.15) and the lower sign in (7.16) correspond to $\eta = 1$, and the opposite signs correspond to $\eta = -1$. Thus, for each space V_+ or V_-, a general solution of the system has $4(2s + 1)$ independent components and carries a reducible representation of the improper Poincaré group. This representation splits into four irreps labelled by different signs of η and p_0.

The formulation under consideration allows the coupling of higher spins with an external electromagnetic field. Indeed, unlike Eq. (7.8), the system (7.14) contains the only one equation with space-time derivatives ∂_μ. (Notice that the first equation of (7.8) is a consequence of other two ones only for $s = 1/2$ and $s = 1$ (Gitman and Shelepin, in press), i.e., for Dirac and Duffin–Kemmer equations). Particles with definite spin s and mass m are described by Klein–Gordon equation with polarization,

$$\left[(\hat{p} - eA)^2 - \frac{e}{2s}\hat{S}^{\mu\nu}F_{\mu\nu} - m^2\right]\psi(x) = 0,$$

where $\psi(x)$ carries the representation $(s0) \oplus (0s)$ of the Lorentz group (Feynman and Gell-Mann, 1958; Hurley, 1971, 1974; Ionesco-Pallas, 1967; Kruglov, Preprint hep-ph/9908410). For $s = 1/2$, this equation is the squared Dirac equation. Solutions of the Klein–Gordon equation with polarization are casual, they have $4(2s + 1)$ independent components (for any sign of energy there are solutions with both signs of the intrinsic parity $\eta = \pm 1$), two times more components than solutions of Dirac and Duffin–Kemmer equations.

We have seen that in the massive case, the transformations P and T_{w} map any irrep of the improper Poincaré group into itself. The operator P labels irreps of the improper Poincaré group. Wigner time reversal T_{w} corresponds to the reversal of the direction of motion and does not change characteristics of representations of the Poincaré group extended by other discrete transformations (η and signs of energy and PCT_{w}-charge). For example, for spin-1/2 particles at the rest frame (see (7.1)), we have $e^{imx^0}z^\alpha \overset{T_{\mathrm{w}}}{\to} e^{imx^0}z_\alpha$, and the transformation T_{w} reduces to the rotation by the angle π. In the general case, T_{w} is not reduced to some continuous or discrete transformations. The transformation changes signs both of momentum vector and spin pseudovector, whereas P changes signs of the momentum vector only.

Two discrete transformations interchange nonequivalent representations of the extended Poincaré group. As such transformations one can choose I_x and I_z

or I_x and C, as it is done below, where the first sign is one of $PCT_{\rm w}$-charge and the second sign is one of p_0:

$$
\begin{array}{cccccccc}
 & ++ & & +- & ++ & & +- & \\
C \updownarrow & & \eta = 1 & & & \eta = -1 & & \updownarrow C \\
 & -- & & -+ & -- & & -+ & \\
 & & \overset{\leftrightarrow}{I_x} & & & \overset{\leftrightarrow}{I_x} & &
\end{array}
$$

Let us touch the problem of relative parity of particles and antiparticles. As it was first pointed out by Nigam and Foldy (1956) for spin-$1/2$ case, such problem admits different treatments; discussions for another spins can be found by Ahluwalia (1996), Ahluwalia *et al.* (1993), and Silagadze (1992). Consider the problem in the framework of the representation theory of the extended Poincaré group.

As was mentioned above, the charge conjugation and $PCT_{\rm w}$-transformation cannot change the intrinsic parity η, since C and I_z commute with P. Suppose that a particle is described by an irrep of the improper Poincaré group. Then we may consider two different possibilities: (a) the corresponding antiparticle is described by $PCT_{\rm w}$-conjugate (or charge-conjugate) irrep, in such a case parities of the particle and the antiparticle must coincide for any spin; (b) the corresponding antiparticle is described by an irrep, which is labelled not only by the opposite $PCT_{\rm w}$-charge but also by the opposite parity η. In the latter case, irreps describing particles and antiparticles are not connected by transformations C or $PCT_{\rm w}$.

Usually, the relation between parities of particles and antiparticles is derived from the corresponding wave equations. Consider some relativistic wave equation describing field with definite spin and mass. As a rule, a general solution of a given equation carries a reducible representation of the improper Poincaré group; irreducible subrepresentations (or their charge conjugated) are identified with particle and antiparticle fields. Since different equations have different structure of the solutions, both possibilities mentioned above can be realized in such an approach.

Consider some examples. One can suppose that for $s = 1/2$ “wave function of antiparticle is a bispinor charge-conjugate to some negative frequency solution of the Dirac equation” (Berestetskii *et al.*, 1971). Free Dirac equation has solutions corresponding to two nonequivalent irreps of the improper Poincaré group; these irreps are characterized by opposite signs of η and p_0. If a positive frequency solution has the intrinsic parity η, then negative frequency solution has the opposite intrinsic parity $-\eta$. This parity is not changed under the charge conjugation and intrinsic parities of particles and antiparticles are opposite. Solutions of the Duffin–Kemmer equation with different signs of energy have identical intrinsic parities. Thus, a standard point of view is that the intrinsic parities for spin one particles and antiparticles are the same. However, studying some relativistic wave equations associated with the representations $(s0) \oplus (0s)$ of the Lorentz group, one can conclude that intrinsic parities of particles and antiparticles are opposite for integer spins (Ahluwalia, 1996; Ahluwalia *et al.*, 1993).

8. GROUP-THEORETICAL DERIVATION OF THE DIRAC EQUATION

Let us consider a pure group-theoretical derivation of the Dirac equation in detail. (An heuristic discussion of the problem can be found by Ahluwalia (1996), Ahluwalia *et al.* (1993), Gaioli and Alvarez (1995), and Ryder (1988).) This derivation is based on the construction of the extended Poincaré group representations with some fixed characteristics. In addition to the evident conditions (fixing the mass and spinor representation of the Lorentz group) it is necessary to demand that states with definite energy possess definite parity, and also that the states possess definite $PCT_{\rm w}$-charge. As we will show, the sign of mass term in the Dirac equation coincides with the sign of the product of three characteristics of the extended Poincaré group representations, namely, the intrinsic parity, the sign of $PCT_{\rm w}$-charge, and the sign of energy. Notice that the consideration and attempts of physical interpretation of two possible signs of the mass term in the Dirac equation have a long history (see, in particular, Barut and Ziino, 1993; Brana and Ljolje, 1980; Dvoeglazov, 1996; Markov, 1964, and references therein).

Consider a representation of the extended Poincaré group with the following characteristics: (i) definite mass $m > 0$; (ii) definite $PCT_{\rm w}$-charge; (iii) states with definite sign of energy possess definite intrinsic parity η and vice versa; and (iv) fields $f(x, \mathbf{z})$ with above characteristics is linear in $\mathbf{z}$ (the latter corresponds to fixing the representation $(\frac{1}{2}\ 0) \oplus (0\ \frac{1}{2})$ of the spin Lorentz subgroup).

According to (iii), this reducible representation of the extended Poincaré group, which we denote by T_D, is the direct sum of two representations with the opposite signs of energy and intrinsic parity.

The suppositions (ii) and (iv) allow the only scalar functions of the form

$$f_+(x, \mathbf{z}) = z\psi_{\rm R} + \overset{*}{\underline{z}}\,\psi_{\rm L}, \quad f_-(x, \mathbf{z}) = \underline{z}\psi_{\rm R} + \overset{*}{z}\,\psi_{\rm L}, \tag{8.1}$$

where we have introduced columns $\psi_{\rm L} = (\psi^{\dot\alpha})$, $\psi_{\rm R} = (\psi_\alpha)$. These functions correspond to two possible signs of $PCT_{\rm w}$-charge. According to (i), there exist functions $f(x, \mathbf{z})$ corresponding to particles in the rest frame such that $\hat p_0 f(x, \mathbf{z}) = p_0 f(x, \mathbf{z})$, $\hat p_k f(x, \mathbf{z}) = 0$, where the energy $p_0 = \varepsilon_{\rm E} m$, $\varepsilon_{\rm E} = {\rm sign}\, p_0$. According to (iii), these functions are characterized by the parity η, defined as the eigenvalue of the space inversion operator, $P f(x, \mathbf{z}) = \eta f(x, \mathbf{z})$. Using the latter equation and the relation (3.28), we obtain: $\psi_{\rm R}(\mathring{p}) = -\eta\psi_{\rm L}(\mathring{p})$ for the functions $f_+(x, \mathbf{z})$, and $\psi_{\rm R}(\mathring{p}) = \eta\psi_{\rm L}(\mathring{p})$ for the functions $f_-(x, \mathbf{z})$ where $\mathring{p} = (\varepsilon_{\rm E} m, 0)$. Both the cases can be described by one equation

$$\psi_{\rm R}(\mathring{p}) = \varepsilon_{\rm c}\eta\psi_{\rm L}(\mathring{p}), \tag{8.2}$$

where $\varepsilon_{\rm c} = {\rm sign} S_3^{\rm R}$ is the sign of the charge.

The Lorentz transformation of the spinors $\psi_R(p) = U\psi_R(\overset{\circ}{p})$, $\psi_L(p) = (U^\dagger)^{-1}\psi_L(\overset{\circ}{p})$ results in a transition to a state, which is characterized by the momentum $P = UP_0U^\dagger$, where $P = p_\mu\sigma^\mu$, $P_0 = \varepsilon_E m\sigma_0$. Thus, we obtain

$$\varepsilon_E mUU^\dagger = p_\mu\sigma^\mu. \tag{8.3}$$

Taking into account the transformation law for spinors, we can rewrite (8.2) in the form

$$\psi_R = \varepsilon_c\eta UU^\dagger\psi_L, \quad \psi_L = \varepsilon_c\eta(UU^\dagger)^{-1}\psi_R.$$

Using (8.3), we can express $UU^\dagger$ in terms of the momentum,

$$m\psi_R = \varepsilon_c\varepsilon_E\eta p_u\sigma^\mu\psi_L, \quad m\psi_L = \varepsilon_c\varepsilon_E\eta p_\mu\bar{\sigma}^\mu\psi_R,$$

and combine these two equations into the one

$$(p_\mu\gamma^\mu - \varepsilon_c\varepsilon_E\eta m)\Psi = 0, \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}. \tag{8.4}$$

Finally, for plane waves, one can change the momentum p_μ by the corresponding operator $\hat{p}_\mu$. Since the plane waves form a basis of the representation T_D and the superposition principle holds, the states belonging to T_D are subjected to the equation

$$(\hat{p}_\mu\gamma^\mu - \varepsilon_c\varepsilon_E\eta m)\Psi = 0. \tag{8.5}$$

In the above consideration, we could use a more restrictive condition of irreducibility of the representation of the improper Poincaré group instead of (iii). But, in any case, general solution of the equation obtained includes states with both signs of intrinsic parity and energy and carry reducible representation obeying the condition (iii). The above consideration also shows the impossibility of the derivation of the Dirac equation only in terms of the proper or improper Poincaré group representations, since the Dirac equation connects signs of the energy ε_E, of the parity η, and of the charge ε_c, which characterize representations of the extended Poincaré group.

9. DISCRETE SYMMETRIES OF RELATIVISTIC WAVE EQUATIONS. MASSLESS CASE

For spin-tensor massless fields with integer and half-integer spins, eigenvalues of the Casimir operators $\hat{p}^2$ and $\hat{W}^2$ are zero (see, e.g., Tung, 1985). Such fields obey the conditions

$$\hat{W}_\mu f(x, \mathbf{z}) = \lambda\hat{p}_\mu f(x, \mathbf{z}), \tag{9.1}$$

where λ is the helicity. In particular, for $\mu = 0$ we have

$$\hat{\mathbf{p}}\hat{\mathbf{S}} f(x, \mathbf{z}) = \lambda\hat{p}_0 f(x, \mathbf{z}). \tag{9.2}$$

The transformations P and C change the sign in Eq. (9.2); on the other hand, the transformations I_x, I_z, $T_{\rm sch}$, which change the sign of mass term in the Dirac equation, are symmetry transformations of the Eq. (9.2). Discrete symmetries of the Eq. (9.2) are generated by three independent operations. For example, these could be I_x, CP, $T_{\rm w}$, the first of which is not a symmetry transformation for the Dirac equation.

The Weyl equations $\hat{\mathbf{p}}\sigma\Psi(x) = \pm\hat{p}^0\Psi(x)$ are particular cases of Eq. (9.2) for helicities $\pm 1/2$, respectively; these equations can be obtained by the substitution of the function $f(x,z) = \Psi_\alpha(x)z^\alpha$ into (9.2).

Massless irreps of the proper Poincaré group are labelled by two numbers (see Table IV), namely, by the helicity $\lambda = \mathbf{pS}/p_0$ and by p_0 sign. If it is not necessary to consider states with definite parity, then instead of V_+ and V_- it is natural to consider four subspaces of functions $f(x,z)$, $f(x,\underline{z})$, $f(x,\overset{*}{z})$, $f(x,\overset{*}{\underline{z}})$.

In each subspace the Eq. (9.2) has four solutions with definite chirality s. These solutions describe a motion along the axis x^3 and are labelled by signs of the helicity and p_0. Considering the action of the operators C and I_z on the solutions, we can see that these solutions describe particles that do not coincide with their antiparticles.

For particles with $p_0 > 0$ we have

$$\lambda = s : e^{i(px^0+px^3)}(z^1)^{2s}, \quad e^{i(px^0+px^3)}(\overset{*}{\underline{z}}_{\dot 1})^{2s}, \tag{9.3}$$

$$\lambda = -s : e^{i(px^0+px^3)}(z^2)^{2s}, \quad e^{i(px^0+px^3)}(\overset{*}{\underline{z}}_{\dot 2})^{2s}, \tag{9.4}$$

and for antiparticles with $p_0 > 0$

$$\lambda = s : e^{i(px^0+px^3)}(\underline{z}^1)^{2s}, \quad e^{i(px^0+px^3)}(\overset{*}{z}_{\dot 1})^{2s}, \tag{9.5}$$

$$\lambda = -s : e^{i(px^0+px^3)}(\underline{z}^2)^{2s}, \quad e^{i(px^0+px^3)}(\overset{*}{z}_{\dot 2})^{2s}. \tag{9.6}$$

The operators P and C interchange states with opposite chirality. The operator I_z, interchanging the states with opposite $PCT_{\rm w}$-charge, does not change signs of the chirality and of the energy. The signs of the helicity and of the chirality are changed simultaneously under the discrete transformations.

Above we have developed the description of particles which differ from their antiparticles. Let us consider as an example the description of pure neutral massless spin-1 particles (let say photons) in terms of a scalar field on the Poincaré group. Such a particle coincides with its antiparticle (it has zero $PCT_{\rm w}$-charge) and has the chirality ± 1. Quadratic in $\mathbf{z} = (z, \underline{z}, \overset{*}{z}, \overset{*}{\underline{z}})$ functions that obey these conditions depend on $z^\alpha\underline{z}^\beta$, $\overset{*}{z}_{\dot\alpha}\overset{*}{\underline{z}}_{\dot\beta}$ only and must be zero vectors for $\hat{S}_3^{\rm R}$. Thus, pure neutral massless spin-1 particles are described by scalar functions of the form

$$f(x,\mathbf{z}) = \chi_{\alpha\beta}(x)z^\alpha\underline{z}^\beta + \psi^{\dot\alpha\dot\beta}(x)\overset{*}{z}_{\dot\alpha}\overset{*}{\underline{z}}_{\dot\beta} = \frac{1}{2}F_{\mu\nu}(x)q^{\mu\nu}, \tag{9.7}$$

where

$$q_{\mu\nu} = -q_{\nu\mu} = \frac{1}{2}((\sigma_{\mu\nu})_{\alpha\beta} z^{\alpha} \underline{z}^{\beta} + (\bar{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}} \overset{*}{z}{}^{\dot{\alpha}} \overset{*}{\underline{z}}{}^{\dot{\beta}}), \quad \overset{*}{q}_{\mu\nu} = q_{\mu\nu}, \tag{9.8}$$

$$F_{\mu\nu}(x) = -2((\sigma_{\mu\nu})_{\alpha\beta} \chi^{\alpha\beta}(x) + (\bar{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}} \psi^{\dot{\alpha}\dot{\beta}}(x)). \tag{9.9}$$

The functions $\chi_{\alpha\beta}(x)$ and $\psi_{\dot{\alpha}\dot{\beta}}(x)$ must be symmetric in their indices; otherwise by virtue of the constraint $z^1\underline{z}^2 - z^2\underline{z}^1 = 1$ (which is a consequence of the unimodularity of $SL(2, C)$) the field (9.7) can contain components $\chi_{[\alpha\beta]}(x)$ and $\psi_{[\dot{\alpha}\dot{\beta}]}(x)$ of zero spin. Therefore, formulations in terms of $\chi_{\alpha\beta}(x)$, $\psi_{\dot{\alpha}\dot{\beta}}(x)$, and $F_{\mu\nu}(x)$ are equivalent. Left and right fields can be described by the functions

$$F^{\mathrm{L}}_{\mu\nu}(x) = F_{\mu\nu}(x) - i\tilde{F}_{\mu\nu}(x) = -4(\bar{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}} \psi^{\dot{\alpha}\dot{\beta}}(x), \tag{9.10}$$

$$F^{\mathrm{R}}_{\mu\nu}(x) = F_{\mu\nu}(x) + i\tilde{F}_{\mu\nu}(x) = -4(\sigma_{\mu\nu})_{\alpha\beta} \chi^{\alpha\beta}(x), \tag{9.11}$$

where $\tilde{F}_{\mu\nu}(x) = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$.

To describe states with a definite helicity, the functions (9.7) should obey the equation (9.2) for $\lambda = \pm 1$,

$$(\hat{\mathbf{p}}\hat{\mathbf{S}} \mp \hat{p}_0) f(x, \mathbf{z}) = 0. \tag{9.12}$$

For $p_0 > 0$, the Eq. (9.12) has four solutions which correspond to a motion along the axis x^3. These solutions differ by signs of helicity and chirality:

$$\lambda = 1 : e^{i(px^0 + px^3)} z^1 \underline{z}^1, \quad e^{i(px^0 + px^3)} \overset{*}{z}_{\dot{1}} \overset{*}{\underline{z}}_{\dot{1}}, \tag{9.13}$$

$$\lambda = -1 : e^{i(px^0 + px^3)} z^2 \underline{z}^2, \quad e^{i(px^0 + px^3)} \overset{*}{z}_{\dot{2}} \overset{*}{\underline{z}}_{\dot{2}}. \tag{9.14}$$

Fixing the relative sign between helicity and chirality (this sign distinguishes the equivalent representations of the extended Poincaré group), we obtain two solutions corresponding to two polarization states.

Substituting the functions $f_{\mathrm{L}}(x, \mathbf{z}) = \psi^{\dot{\alpha}\dot{\beta}}(x) \overset{*}{z}_{\dot{\alpha}} \overset{*}{\underline{z}}_{\dot{\beta}}$ and $f_{\mathrm{R}}(x, \mathbf{z}) = \chi_{\alpha\beta}(x) z^{\alpha} \underline{z}^{\beta}$ into (9.12) (for $\lambda = \pm 1$ respectively) and going over to the vector notation in accordance with (9.10) and (9.11), we obtain equations for $F^{\mathrm{L}}_{\mu\nu}(x)$ and $F^{\mathrm{R}}_{\mu\nu}(x)$,

$$\partial^{\mu} F^{\mathrm{L}}_{\mu\nu}(x) = 0, \quad \partial^{\mu} F^{\mathrm{R}}_{\mu\nu}(x) = 0. \tag{9.15}$$

Obviously they are equivalent to the Maxwell equations

$$\partial^{\mu} F_{\mu\nu}(x) = 0, \quad \partial^{\mu} \tilde{F}_{\mu\nu}(x) = 0. \tag{9.16}$$

As is known, the second equation results in $F_{\mu\nu}(x) = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$, where complex potentials A_{μ} are introduced.

Taking into account the action of discrete transformations on $\mathbf{z}$ (see (3.28)), we find $q^{\mu\nu} \stackrel{P}{\to} (-1)^{\delta_{0\mu}+\delta_{0\mu}} q^{\mu\nu}$, $q^{\mu\nu} \stackrel{I_z}{\to} -q^{\mu\nu}$, $q^{\mu\nu} \stackrel{C}{\to} q^{\mu\nu}$. Then as a consequence of (9.7), we obtain

$$P : F_{\mu\nu}(x) \to (-1)^{\delta_{0\mu}+\delta_{0\mu}} F_{\mu\nu}(\bar{x}), \quad A_\mu(x) \to -(-1)^{\delta_{0\mu}} A_\mu(\bar{x}); \tag{9.17}$$

$$I_z : F_{\mu\nu}(x) \to -F_{\mu\nu}(x), \quad A_\mu(x) \to -A_\mu(x), \tag{9.18}$$

$$C : F_{\mu\nu}(x) \to \overset{*}{F}_{\mu\nu}(x), \quad A_\mu(x) \to \overset{*}{A}_\mu(x). \tag{9.19}$$

It is easy to see that the C transformation, acting on the functions (9.7) as a complex conjugation, interchanges states with opposite helicities. Thus C transformation cannot be considered separately for left and right fields (see, e.g., Ohnuki, 1988). The transformation I_3 does not changes the functions (9.7).

In contrast to the initial Eq. (9.12), where p_0 sign is changed under the space reflection and the charge conjugation, the Eq. (9.16) are invariant under the latter transformations since left and right fields enter in $F_{\mu\nu}(x)$ on an equal footing. Thus, P, I_x, C, T_{w} are symmetry transformations for the Eq. (9.16).

We can consider real and imaginary parts of $F_{\mu\nu}(x)$ as two independent real fields $F^{(1)}_{\mu\nu}(x)$ and $F^{(2)}_{\mu\nu}(x)$; they satisfy the same Eq. (9.16) and are characterized by opposite parities with respect to the charge conjugation operation. However, these fields do not describe states with a definite helicity since they include both left and right components according to (9.9). One ought to notice that $F^{\mathrm{L}}_{\mu\nu}(x)$ and $F^{\mathrm{R}}_{\mu\nu}(x)$ cannot be treated as classical electromagnetic fields, but can be treated as wave functions of left-handed and right-handed photons (Akhiezer and Berestetskii, 1981; Białynicki-Birula, 1994; Ohnuki, 1988).

10. CONCLUSION

We have shown that the representation theory of the proper Poincaré group implies the existence of five nontrivial independent discrete transformations corresponding to involutory automorphisms of the group. As such transformations one can choose space reflection P, inversion I_x, charge conjugation C, Wigner time reversal T_{w}. The fifth transformation for the most fields of physical interest (except the Majorana field) is reduced to the multiplication by a phase factor.

Considering discrete automorphisms as operators acting in the space of the functions on the Poincaré group, we have obtained the explicit form for the discrete transformations of arbitrary spin fields without any appealing to relativistic wave equations. The examination of the action of automorphisms on the operators, in particular, on the generators of the Poincaré group, ensures the possibility to get transformation laws of corresponding physical quantities. The analysis of the scalar field on the group allows us to construct explicitly the states corresponding to representations of the extended Poincaré group, and also to give the classification of the solutions of various types of relativistic wave equations with respect to representations of the extended group.

Since in the general case a relativistic wave equation can fix some characteristics of the extended Poincaré group representation, which are changed under the discrete transformations, only a part of the discrete transformations forms symmetry transformations of the equation. In particular, discrete symmetries of the Dirac equation and of the Weyl equation are generated by two different sets of the discrete transformations operators, P, C, T_{w} and PC, I_x, T_{w} respectively.

Being based on the concept of the field on the group and on the consideration of the group automorphisms, the approach developed can be applied to the analysis of discrete symmetries in other dimensions and also to other space-time symmetry groups.

APPENDIX A: THE LEFT AND RIGHT GENERATORS OF *SL*(2, *C*) IN THE SPACE OF SCALAR FUNCTIONS ON THE POINCARÉ GROUP

The left and right spin operators have the form (Gitman and Shelepin, in press)

$$\hat{S}_k = \frac{1}{2}\left(z\sigma_k\partial_z - \overset{*}{z}\overset{*}{\sigma}_k\partial_{\overset{*}{z}}\right) + \cdots,$$

$$\hat{B}_k = \frac{i}{2}\left(z\sigma_k\partial_z + \overset{*}{z}\overset{*}{\sigma}_k\partial_{\overset{*}{z}}\right) + \cdots, \quad z = (z^1\, z^2), \quad \partial_z = (\partial/\partial z^1 \partial/\partial z^2)^T; \tag{A1}$$

$$\hat{S}^{\mathrm{R}}_k = -\frac{1}{2}\left(\chi\overset{*}{\sigma}_k\partial_\chi - \overset{*}{\chi}\sigma_k\partial_{\overset{*}{\chi}}\right) + \cdots, \quad \chi = (z^1\, \underline{z}^1),$$

$$\hat{B}^{\mathrm{R}}_k = -\frac{i}{2}(\chi\overset{*}{\sigma}_k\partial_\chi + \overset{*}{\chi}\,/\partial z^1\partial/\partial\underline{z}^1)^T; \tag{A2}$$

By three dots we have denoted here expressions obtained from the preceding ones by the substitution $z \to \underline{z} = (\underline{z}^1\, \underline{z}^2)$, $\chi \to \chi' = (z^2\, \underline{z}^2)$. Two first equations can be rewritten as

$$\hat{S}^{\mu\nu} = \frac{1}{2}((\sigma^{\mu\nu})_\alpha{}^\beta z^\alpha\partial_\beta + (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\overset{*}{\underline{z}}_{\dot{\alpha}}\underline{\partial}^{\dot{\beta}}) - \text{c.c.}, \tag{A3}$$

where $\partial_\alpha = \partial/\partial z^\alpha$, $\underline{\partial}^{\dot\alpha} = \partial/\partial \underline{\overset{*}{z}}_{\dot\alpha}$,

$$(\sigma^{\mu\nu})_\alpha{}^\beta = -\frac{i}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)_\alpha{}^\beta, \quad (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = -\frac{i}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu)^{\dot\alpha}{}_{\dot\beta}, \quad \text{(A4)}$$

and c.c. is complex conjugate term.

APPENDIX B: EQUATIONS FOR DEFINITE MASS AND SPIN IN TERMS OF SCALAR FUNCTIONS ON THE POINCARÉ GROUP

The equations for scalar functions $f(x, z, \underline{\overset{*}{z}})$ on the Poincaré group

$$\hat p^2 f(x, z, \underline{\overset{*}{z}}) = m^2 f(x, z, \underline{\overset{*}{z}}), \quad \text{(B1)}$$

$$\hat p_\mu \hat\Gamma^\mu f(x, z, \underline{\overset{*}{z}}) = ms f(x, z, \underline{\overset{*}{z}}), \quad \text{(B2)}$$

where

$$\hat\Gamma^\mu = \frac{1}{2}(\bar\sigma^{\mu\dot\alpha\alpha}\underline{\overset{*}{z}}_{\dot\alpha}\partial_\alpha + \sigma^\mu{}_{\alpha\dot\alpha} z^\alpha \underline{\partial}^{\dot\alpha}) - \text{c.c.} \quad \text{(B3)}$$

describe a particle with fixed mass $m > 0$ and spin s, if we suppose that $f(x, z, \underline{\overset{*}{z}})$ is a polynomial of the power $2s$ in z, $\underline{\overset{*}{z}}$ (Gitman and Shelepin, in press). Analogous statement also holds for polynomial in $\underline{z}$, $\overset{*}{z}$ functions $f(x, \underline{z}, \overset{*}{z})$. Operators $\hat\Gamma^\mu$ and $\hat S^{\mu\nu}$ obey $SO(3, 2)$ group commutation relations

$$[\hat S^{\lambda\mu}, \hat\Gamma^\nu] = i(\eta^{\mu\nu}\hat\Gamma^\lambda - \eta^{\lambda\nu}\hat\Gamma^\mu), \quad [\hat\Gamma^\mu, \hat\Gamma^\nu] = -i\hat S^{\mu\nu}. \quad \text{(B4)}$$

These are commutation relations for the matrices $\gamma^\mu/2$. Together with the chirality operator

$$\hat\Gamma^5 = \frac{1}{2}(z^\alpha\partial_\alpha - \underline{\overset{*}{z}}_{\dot\alpha}\underline{\partial}^{\dot\alpha}) - \text{c.c.}, \quad \text{(B5)}$$

and the operators $\underline{\hat\Gamma}^\mu = i[\hat\Gamma^\mu, \hat\Gamma^5]$, $\hat S_3^{\mathrm{R}}$, the operators $\hat\Gamma^\mu$, $\hat S^{\mu\nu}$ form a set of 16 operators, which do not change the power of polynomials $f(x, z, \underline{\overset{*}{z}})$ in z, $\underline{\overset{*}{z}}$.

Being written in spin-tensor notation, the equation (B2) for $s = 1/2$ appears to be the Dirac equation and for $s = 1$ the Duffin–Kemmer equation. In the general case, being written in spin-tensor notation, the system (B1)–(B2) consists of the Klein–Gordon equation and symmetric Bhabha equation (Gitman and Shelepin, 2001). This system is equivalent to the Bargmann–Wigner equations (Gitman and Shelepin, 2001; Loide *et al.*, 1997).

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